Floer theory

Will J. Merry

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Disclaimer: These notes have not yet been proofread *at all*. They are probably littered with errors. Please let me know[1] of any you find!

Contents

[1] By email to merry@math.ethz.ch. I am very grateful to C. Antony for pointing out many typos and errors.

CHAPTER 1

Morse theory

In this first lecture we will give a rapid overview of Morse theory from a 'modern' viewpoint. For simplicity here we stick to Morse functions on finite dimensional closed manifolds, but with appropriate modifications a lot of what follows works in either the non-compact or infinite dimensional setting. A key textbook reference for Morse theory is the book of Schwarz [Sch93]. In this section there are essentially no proofs. Most of this course will be concerned with establishing similar theorems in the Floer setup. These will be proved in full. *Warning: If this material is not at least vaguely familiar then you are probably better off dropping the course now! It is only going to get harder...*

In a lot of what follows we will talk about *Banach manifolds* and *Banach bundles* over them. These are not as scary as they seem! A *Banach manifold* $\mathcal{B}$ is defined in exactly the same way as a normal manifold B is, the only difference is that $\mathcal{B}$ should locally look like some fixed Banach space X rather than a fixed Euclidean space $\mathbb{R}^n$. Since $\mathbb{R}^n$ is itself a Banach space, normal manifolds are of course a special case of Banach manifolds. The key difference between Banach manifolds and normal manifolds is that the latter may be infinite dimensional.

Exercise 1.1. Look up how much of the differential geometry you know and love still applies for Banach manifolds. Feel both reassured and scared at the same time.

Similarly a *Banach bundle* $\pi : \mathcal{E} \to \mathcal{B}$ over a Banach manifold is defined in exactly the same way as a vector bundle $\pi : E \to B$ over a normal manifold, the only difference being that each fibre $\mathcal{E}_x := \pi^{-1}(x)$ should look like a Banach space X rather than a vector space V.

Definition 1.2. Let $\pi : \mathcal{E} \to \mathcal{B}$ denote a Banach bundle and $\sigma \in \Gamma(\mathcal{B}, \mathcal{E})$ a smooth section. Denote by

$$\mathcal{M}_\sigma := \{x \in \mathcal{B} \mid \sigma(x) = 0_x \in \mathcal{E}_x\}.$$

If $x \in \mathcal{M}_\sigma$ then the *vertical derivative* $D^v\sigma(x)$ is a map

$$D^v\sigma(x) : T_x\mathcal{B} \to \mathcal{E}_x$$

defined as follows. Firstly, the normal differential $D\sigma(x)$ is a map $D\sigma(x) : T_x\mathcal{B} \to T_{\sigma(x)}\mathcal{E} = T_{0_x}\mathcal{E}$. Next, if $\mathfrak{o} \in \Gamma(\mathcal{B}, \mathcal{E})$ denotes the zero section then one can write

$$T_{0_x}\mathcal{E} = T_{0_x}(\mathfrak{o}(\mathcal{B})) \oplus T_{0_x}\mathcal{E}_x \cong T_x\mathcal{B} \oplus \mathcal{E}_x.$$

The vertical derivative is then the composition of $D\sigma(x)$ and the projection $\mathrm{proj} : T_{0_x}\mathcal{E} \to \mathcal{E}_x$.

Definition 1.3. Suppose $x \in \mathcal{M}_\sigma$. We call x a *regular zero* if the vertical derivative $D^v\sigma(x) : T_x\mathcal{B} \to \mathcal{E}_x$ is a surjective linear map between the Banach spaces $T_x\mathcal{B}$ and $\mathcal{E}_x$. This is equivalent to asking that

$$D^v\sigma(x)[T_x\mathcal{B}] \oplus D\mathfrak{o}(x)[T_x\mathcal{B}] = T_{0_x}\mathcal{E},$$

i.e., that σ *is transverse to the zero section* $\mathfrak{o}$ *at* x.

In finite dimensions, one has:

Theorem 1.4 (*Finite dimensional implicit function theorem*)**.** *Suppose $\pi : E^{n+k} \to B^n$ is a vector bundle of rank k, and $\sigma \in \Gamma(B,E)$ is a section. Assume that the vertical derivative $D^v\sigma(x)$ is regular at every point $x \in M_\sigma := \sigma^{-1}(0)$. Then M_σ is a submanifold of B of dimension $n-k$.*

In the infinite dimensional case a similar result is true. In order to state the result precisely, let us recall the definition of a *Fredholm map*.

Definition 1.5. Suppose $T \in \mathsf{L}(X,Y)$ is a bounded linear map between two Banach spaces X and Y. Then $\ker T$ and $\operatorname{coker} T := Y/\operatorname{ran} T$ are linear subspaces of X and Y respectively. We say that T is a *Fredholm* operator if both $\ker T$ and $\operatorname{coker} T$ are finite dimensional, and if this is the case we define the *Fredholm index* of T to be the integer

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T.$$

Theorem 1.6 (*Infinite dimensional implicit function theorem*)**.** *Suppose $\pi : \mathcal{E} \to \mathcal{B}$ is a Banach bundle and $\sigma \in \Gamma(\mathcal{B},\mathcal{E})$. Assume that the vertical derivative $D^v\sigma(x)$ is Fredholm, and in addition is surjective and admits a right inverse for every $x \in \mathcal{M}_\sigma := \sigma^{-1}(0)$. Then $\mathcal{M}_\sigma$ admits the structure of a submanifold of $\mathcal{B}$ of (finite) dimension equal to the Fredholm index of $D^v\sigma(x)$.*

Exercise 1.7. Check that the finite dimensional implicit function theorem is a corollary of the infinite dimensional implicit function theorem.

Definition 1.8. In general, a space $\mathcal{M}$ is said to have *virtual dimension* $k \in \mathbb{Z}$, written

$$\operatorname{virdim} \mathcal{M} = k,$$

if $\mathcal{M}$ can be seen as the set of zeros of a smooth section of a Banach bundle over a Banach manifold, whose vertical derivative is Fredholm and of index k. When such a section is transverse to the zero section (and hence every zero is regular), Theorem 1.6 implies that either $\mathcal{M}$ is empty, or $k \geq 0$ and $\mathcal{M}$ admits the structure of a smooth manifold of dimension k.

Remark 1.9. Typical proofs in Floer theory go along the following lines: suppose we want to prove that 'bad property X' can never happen. We first show that the space of states where X happens has virtual dimension $k < 0$. Then we show that the section defining this space can be made transverse. This implies that X can never happen.

We will now give two applications to this material, both related to Morse theory. The first is finite dimensional, whereas the second will be infinite dimensional.

Definition 1.10. Let B denote a smooth (finite dimensional) closed manifold and fix $f \in C^\infty(B)$. Then $df \in \Gamma(B, T^*B)$, and in this case

$$M_{df} = \operatorname{crit}(f) = \{x \in B \mid df_x = 0\}.$$

One says that $x \in \operatorname{crit}(f)$ is a *non-degenerate* critical point if x is a regular zero of the vertical derivative $D^v(df)(x)$. One says that f is a *Morse function* if every critical point of f is non-degenerate.

Exercise 1.11. Suppose $(x_1, \dots, x_n)$ are local coordinates at x. Identify $D^v(df)(x)$ with the Hessian matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_x\right]_{i,j}.$$

Conclude that this is the same definition of Morse function that you've known since you were five.

Now fix a Riemannian metric g on B. We denote by ∇f or $\nabla_g f$ the *gradient* of f with respect to g, which is the unique vector field $\nabla f \in \Gamma(B, TB)$ such that

$$g(\nabla f(x), \xi) = df(x)[\xi], \quad \text{for all } x \in B,\ \xi \in T_x B.$$

We denote by

$$\exp_x : D_x \subset T_x B \to U_x \subset B$$

the exponential map (with respect to some Riemannian metric, not necessarily equal to g) on B. Now fix two distinct critical points x, y of f. Let $\mathcal{B}(x, y)$ denote the set of maps $u \in W^{1,2}(\mathbb{R}, B)$ that satisfy

$$\lim_{s \to -\infty} u(s) = x, \quad \lim_{s \to +\infty} u(s) = y,$$

and such that there exists $s_0 > 0$ and $\zeta \in W^{1,2}((-\infty, -s_0], D_x)$ and $\xi \in W^{1,2}([s_0, +\infty), D_y)$ such that

$$u(s) = \exp_x(\zeta(s)), \quad \text{for all } s \in (-\infty, -s_0],$$
$$u(s) = \exp_x(\xi(s)), \quad \text{for all } s \in [s_0, +\infty).$$

Theorem 1.12. *The space $\mathcal{B}(x, y)$ can be given the structure of a Banach manifold. Moreover if $u \in \mathcal{B}(x, y)$ then $T_u\mathcal{B}(x, y)$ can be identified with the space of sections $\xi \in W^{1,2}(\mathbb{R}, u^*TB)$.*

Exercise 1.13. Look in Appendix A of Schwarz's book [Sch93] and study the proof of Theorem 1.12 in depth. Suppress the desire to kill yourself.

We now define a Banach bundle $\pi : \mathcal{E} \to \mathcal{B}(x, y)$ by setting

$$\mathcal{E}_u := L^2(\mathbb{R}, u^*TB), \quad u \in \mathcal{B}(x, y).$$

As with Theorem 1.12, the proof that $\mathcal{E}$ really is a Banach bundle can be found in Schwarz' book [Sch93].

Definition 1.14. Fix $x, y \in \mathrm{crit}(f)$. Define a section $\sigma = \sigma_g \in \Gamma(\mathcal{B}(x, y), \mathcal{E})$ by setting

$$\sigma(u) := \partial_s u + \nabla_g f(u).$$

We will prove the Floer-theoretic version of the next theorem in great detail later on in the course.

Theorem 1.15 (*Properties of σ*)**.**

1. *The section σ is smooth.*
2. *If $u \in \mathcal{M}(x, y) := \sigma^{-1}(0)$ then u is a smooth map.*
3. *The section σ is Fredholm.*

The proof of (1) is somewhat tedious. The proof of (2) is very straightforward, and proceeds by induction. Suppose $u \in W^{1,2}$ satisfies $\sigma(u) = 0$. Now $u \in W^{1,2}$ implies that $\nabla f(u) \in W^{1,2}$, and hence $\partial_s u = -\nabla f(u)$ also belongs to $W^{1,2}$. But this implies that $u \in W^{2,2}$. Iterating this argument, we see $u \in W^{k,2}$ for each $k \in \mathbb{Z}$, and hence $u \in C^\infty$ by the Sobolev embedding theorem. This process (which is far less trivial in the Floer case) is known as *elliptic regularity*.

Definition 1.16. If $x \in \operatorname{crit}(f)$ is a non-degenerate critical point then we denote by $\operatorname{ind}_f(x)$ the number (counted with multiplicity) of strictly negative eigenvalues of $D^v(df)(x)$, and call $\operatorname{ind}_f(x)$ the *Morse index* of f at x.

Theorem 1.17. *The Fredholm index of σ is given by*

$$\operatorname{ind}\sigma = \operatorname{ind}_f(x) - \operatorname{ind}_f(y).$$

It follows from Theorem 1.15, Theorem 1.17, and Theorem 1.6 that *if* $D^v\sigma(u)$ was surjective and admitted a right inverse for each $u \in \mathcal{M}(x,y)$ then $\mathcal{M}(x,y)$ would carry the structure of a smooth manifold of (finite) dimension $\operatorname{ind}_f(x) - \operatorname{ind}_f(y)$. Unfortunately this is in general *not* true. Nevertheless, the next best thing is true.

Theorem 1.18. *Let $\mathsf{Met}(B)$ denote the (Frechet) manifold of all Riemannian metrics on B. There exists a subset $\mathcal{R}_f \subset \mathsf{Met}(B)$ of second category such that if $g \in \mathcal{R}_f$ then for any two critical points $x, y \in \operatorname{crit} f$ and any $u \in \mathcal{M}(x,y)$, the operator $D^v\sigma(u)$ is surjective and admits a right inverse.*

In particular, if we start with some given metric g^0 on B and fix $\varepsilon > 0$ then there exists $g \in \mathcal{R}_f$ such that $\left\|g - g^0\right\|_{C^\infty} < \varepsilon$. We say that g is a *Morse-Smale metric* for f if $g \in \mathcal{R}_f$, and we say a pair $(f,g) \in C^\infty(B) \times \mathsf{Met}(B)$ is a *Morse-Smale pair* if f is a Morse function and g is a Morse-Smale metric for f.

From now on we will always assume that (f,g) is a Morse-Smale pair. Thus the moduli spaces $\mathcal{M}(x,y)$ are always manifolds. We now wish to investigate the compactness properties of these spaces. First note that if $x \neq y$ then $\mathcal{M}(x,y)$ is *never* compact. Indeed, it always admits a free $\mathbb{R}$-action via translation: explicitly if $u \in \mathcal{M}(x,y)$ and $s_0 \in \mathbb{R}$, then the curve $s \mapsto u(s+s_0)$ also belongs to $\mathcal{M}(x,y)$. Thus we see that if $x \neq y$ and $\mathcal{M}(x,y) \neq \emptyset$ then it is always at least 1-dimensional. Hence if $\operatorname{ind}_f(x) = \operatorname{ind}_f(y)$ with $x \neq y$ then $\mathcal{M}(x,y) = \emptyset$.

Definition 1.19. We denote by $\underline{\mathcal{M}}(x,y) := \mathcal{M}(x,y)/\mathbb{R}$. Given $u \in \mathcal{M}(x,y)$ we denote by $\underline{u} \in \underline{\mathcal{M}}(x,y)$ the equivalence class. Thus

$$\dim \underline{\mathcal{M}}(x,y) = \operatorname{ind}_f(x) - \operatorname{ind}_f(y) - 1.$$

Suppose that $\operatorname{ind}_f(x) = \operatorname{ind}_f(y) + 1$, so that $\underline{\mathcal{M}}(x,y)$ is a zero-dimensional space. *If* the space $\underline{\mathcal{M}}(x,y)$ was compact, it would be a finite set, and hence we could 'count' it. Luckily, this is indeed the case:

Theorem 1.20 (*Baby compactness*)**.** *If $\operatorname{ind}_f(x) = \operatorname{ind}_f(y) + 1$ then the space $\underline{\mathcal{M}}(x,y)$ is compact, and hence a finite set.*

Definition 1.21. Given $x, y \in \operatorname{crit} f$ with $\operatorname{ind}_f(x) = \operatorname{ind}_f(y) + 1$, we define the number $n(x,y) \in \mathbb{Z}_2$ to be the parity of the finite set $\underline{\mathcal{M}}(x,y)$:

$$n(x,y) := \#_2 \underline{\mathcal{M}}(x,y).$$

Remark 1.22. In this course we will only ever work with $\mathbb{Z}_2$-coefficients.

We can now define the *Morse complex*. Set

$$\operatorname{crit}_k(f) := \{x \in \operatorname{crit}(f) \mid \operatorname{ind}_f(x) = k\}.$$

Definition 1.23. Let

$$\mathrm{CM}_k(f) := \bigoplus_{x \in \mathrm{crit}_k(f)} \mathbb{Z}_2 \langle x \rangle ,$$

and define

$$\partial = \partial_g : \mathrm{CM}_k(f) \to \mathrm{CM}_{k-1}(f)$$

by requiring that

$$\partial \langle x \rangle := \sum_{y \in \mathrm{crit}_{k-1}(f)} n(x,y) \langle y \rangle ,$$

and then extending by linearity.

We would like to prove that $\partial \circ \partial = 0$. This is not remotely obvious, and to do so will require a more sophisticated compactness result than the one in Theorem 1.20.

Definition 1.24. A sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(x,y)$ is said to *converge up to breaking* if there exist

1. critical points $x = x^0, x^1, \dots, x^m = y$ of f,
2. flow lines $u^j \in \mathcal{M}(x^{j-1}, x^j)$ for $1 \le j \le m$,
3. sequences $(s_k^j)_{k\in\mathbb{N}}$ for $1 \le j \le m$ with $s_k^{j-1} < s_k^j$ and with $s_k^j - s_k^{j-1} \to +\infty$ for each $k \in \mathbb{N}$ and each $1 \le j \le m$,

with the following property: For any compact interval $I \subset \mathbb{R}$, after passing to a subsequence, the sequence $u_k(\cdot + s_k^j)$ converges with all derivatives to $u^j(\cdot)$. In this case we say that (u_k) converges to the *broken gradient flow line* $(u^1, \dots, u^m)$ and we write

$$u_k \rightsquigarrow (u^1, \dots, u^m).$$

Definition 1.25. A sequence $(\underline{u}_k) \subset \underline{\mathcal{M}}(x,y)$ *converges to the broken gradient flow line* $(\underline{u}^1, \dots, \underline{u}^m)$, written

$$\underline{u}_k \rightsquigarrow (\underline{u}^1, \dots, \underline{u}^m),$$

if there exist representatives $u_k \in \underline{u}_k$ and $u^j \in \underline{u}^j$ such that $u_k \rightsquigarrow (u^1, \dots, u^m)$.

Note that if there exists a sequence $(\underline{u}_k) \subset \underline{\mathcal{M}}(x,y)$ that converges to a broken flow line $(\underline{u}^1, \dots, \underline{u}^m)$ then necessarily $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) \ge m+1$. Indeed, one has $\underline{u}^j \in \underline{\mathcal{M}}(x^{j-1}, x^j)$ for some critical points $x = x^0, \dots, x^m = y$, and then $\mathrm{ind}_f(x^{j-1}) - \mathrm{ind}_f(x^j) \ge 1$ for each $1 \le j \le m$.

Theorem 1.26 (*Compactness*)**.** *Suppose x, y are critical points of f which satisfy*

$$\mathrm{ind}_f(x) = \mathrm{ind}_f(y) + m + 1$$

for some $m \ge 0$. Then $\underline{\mathcal{M}}(x,y)$ is compact up to m-fold breaking *in the following sense: suppose $(\underline{u}_k) \subset \underline{\mathcal{M}}(x,y)$ has no convergent subsequence. Then there exists a broken gradient trajectory $(\underline{u}^1, \dots, \underline{u}^l)$ for some $l \le m$ such that after passing to a subsequence, $\underline{u}_k \rightsquigarrow (\underline{u}^1, \dots, \underline{u}^l)$. In particular, if $m = 0$ then every sequence in $\underline{\mathcal{M}}(x,y)$ has a convergent subsequence.*

We next state the 'converse' of Theorem 1.26.

Theorem 1.27 (*Gluing*)**.** *Suppose x, y are critical points of f which satisfy*

$$\mathrm{ind}_f(x) = \mathrm{ind}_f(y) + m + 1$$

for some $m \geq 1$. Then if $1 \leq l \leq m$ and $x = x^0, x^1, \ldots, x^l = y$ are critical points of f such that $\mathrm{ind}_f(x^{j-1}) - \mathrm{ind}_f(x^j) \geq 1$ for each j, and $\underline{u}^j \in \underline{\mathcal{M}}(x^{j-1}, x^j)$ are given flow lines, then there exists a sequence $(\underline{u}_k) \subset \underline{\mathcal{M}}(x, y)$ such that $\underline{u}_k$ converges to the broken flow line $(\underline{u}^1, \ldots, \underline{u}^l)$.

Let us explicitly look at the case where $m = 1$, so that $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) = 2$. Let $\overline{\underline{\mathcal{M}}(x, y)}$ denote the *compactification* of $\underline{\mathcal{M}}(x, y)$ obtained by adding in all possible limits to sequences $(\underline{u}_k) \subset \underline{\mathcal{M}}(x, y)$. Then $\overline{\underline{\mathcal{M}}(x, y)}$ is a compact 1-dimensional manifold. It follows from Theorems 1.26 and 1.27 that:

Theorem 1.28. *Suppose x, y are critical points of f which satisfy*

$$\mathrm{ind}_f(x) = \mathrm{ind}_f(y) + 2$$

The boundary $\partial \overline{\underline{\mathcal{M}}(x, y)}$ of the 1-dimensional manifold $\overline{\underline{\mathcal{M}}(x, y)}$ can be identified as:

$$\partial \overline{\underline{\mathcal{M}}(x, y)} = \bigcup_{\mathrm{ind}_f(z) = \mathrm{ind}_f(x) - 1} \underline{\mathcal{M}}(x, z) \times \underline{\mathcal{M}}(z, y).$$

Corollary 1.29. *Suppose x, y are critical points of f which satisfy*

$$\mathrm{ind}_f(x) = \mathrm{ind}_f(y) + 2.$$

Then

$$\sum_{\mathrm{ind}_f(z) = \mathrm{ind}_f(x) - 1} n(x, z) \cdot n(z, y) = 0 \quad \text{modulo } 2.$$

Proof. A compact 1-dimensional manifold with boundary has an even number or boundary points. ∎

Exercise 1.30. Prove this!

Corollary 1.31. $\partial^2 = 0$.

Proof. This is immediate from Corollary 1.29. Indeed, if $x \in \mathrm{crit}_k(f)$ then since ∂ is linear,

$$\begin{aligned}
\partial(\partial \langle x \rangle) &= \partial \left(\sum_{z \in \mathrm{crit}_{k-1}(f)} n(x, z) \langle z \rangle \right) \\
&= \sum_{z \in \mathrm{crit}_{k-1}(f)} n(x, z) \partial \langle z \rangle \\
&= \sum_{z \in \mathrm{crit}_{k-1}(f)} n(x, z) \left(\sum_{y \in \mathrm{crit}_{k-2}(f)} n(z, y) \langle y \rangle \right) \\
&= \sum_{y \in \mathrm{crit}_{k-2}(f)} \underbrace{\left(\sum_{z \in \mathrm{crit}_{k-1}(f)} n(x, z) \cdot n(z, y) \right)}_{=0 \text{ modulo } 2} \langle y \rangle .
\end{aligned}$$

∎

Let us now discuss independence. Suppose $(f^{\pm}, g^{\pm})$ are two Morse-Smale pairs. Choose a homotopy $(f_s, g_s)_{s\in\mathbb{R}}$ such that

$$(f_s, g_s) = \begin{cases} (f^-, g^-), & s \le -T, \\ (f^+, g^+), & s \ge T, \end{cases}$$

for some $T > 0$.

Definition 1.32. Fix $x^{\pm} \in \mathrm{crit}(f^{\pm})$ and define the moduli space $\mathcal{N}(x^-, x^+)$ to consists of all maps $u \in \mathcal{B}(x^-, x^+)$ such that

$$\partial_s u + \nabla_{g_s} f_s(u) = 0.$$

Thus $\mathcal{N}(x^-, x^+)$ is the zero-set of the s-dependent section $\sigma(u) := \partial_s u + \nabla_{g_s} f_s(u)$. As before, we say that the homotopy (f_s, g_s) is *regular* if the vertical derivative $D^v\sigma(u)$ is surjective and admits a right inverse. As in Theorem 1.18, one can show that a generic homotopy is regular. The Fredholm index of σ is again given by $\mathrm{ind}_{f^-}(x^-) - \mathrm{ind}_{f^+}(x^+)$. Note however that this time there is no $\mathbb{R}$ -action on $\mathcal{N}(x^-, x^+)$. Indeed, if $u \in \mathcal{N}(x^-, x^+)$ and $s_0 \in \mathbb{R}$ then typically $s \mapsto u(s + s_0)$ does *not* belong to $\mathcal{N}(x^-, x^+)$. It is therefore reasonable to hope that when $\mathrm{ind}_{f^-}(x^-) = \mathrm{ind}_{f^+}(x^+)$, the space $\mathcal{N}(x^-, x^+)$ is compact, and thus finite. Luckily this is true:

Theorem 1.33. *Suppose* $\mathrm{ind}_{f^-}(x^-) = \mathrm{ind}_{f^+}(x^+)$. *Then the space* $\mathcal{N}(x^-, x^+)$ *is compact.*

As with Theorem 1.20, this is a consequence of a more general compactness theorem. The upshot is that we can define

$$n_-^+(x^-, x^+) := \#_2\mathcal{N}(x^-, x^+),$$

and then define a map

$$\Phi_-^+ : \mathrm{CM}_k(f^-) \to \mathrm{CM}_k(f^+) \tag{1.1}$$

by setting

$$\Phi_-^+ \left\langle x^- \right\rangle := \sum_{x^+ \in \mathrm{crit}_k(f^+)} n_-^+(x^-, x^+) \left\langle x^+ \right\rangle, \tag{1.2}$$

and then extending by linearity. The next step is to show that Φ_-^+ is a chain map: that is,

$$\partial^+ \circ \Phi_-^+ = \Phi_-^+ \circ \partial^-, \tag{1.3}$$

where $\partial^{\pm}$ denotes the boundary operator on $\mathrm{CM}_*(f^{\pm})$. This would imply that Φ_-^+ descends to define a map

$$\phi_-^+ : \mathrm{HM}_*(f^-, g^-) \to \mathrm{HM}_*(f^+, g^+). \tag{1.4}$$

As in the proof of Corollary 1.31, in order to prove (1.3) we need to analyze the boundary of the compactification $\overline{\mathcal{N}(x^-, x^+)}$ of $\mathcal{N}(x^-, x^+)$ when $\mathrm{ind}_{f^-}(x^-) = \mathrm{ind}_{f^+}(x^+) + 1$. This time we have the following analogue of Theorem 1.28.

Theorem 1.34. *Suppose* $x^- \in \mathrm{crit}_k(f^-)$ *and* $x^+ \in \mathrm{crit}_{k-1}(f^+)$. *The boundary* $\partial\overline{\mathcal{N}(x^-, x^+)}$ *of the 1-dimensional manifold* $\overline{\mathcal{N}(x^-, x^+)}$ *can be identified as:*

$$\partial\overline{\mathcal{N}(x^-, x^+)} = \left(\bigcup_{y^- \in \mathrm{crit}_{k-1}(f^-)} \underline{\mathcal{M}}^-(x^-, y^-) \times \mathcal{N}(y^-, x^+) \right) \cup \left(\bigcup_{y^+ \in \mathrm{crit}_k(f^+)} \mathcal{N}(x^-, y^+) \times \underline{\mathcal{M}}^+(y^+, x^+) \right),$$

where $\underline{\mathcal{M}}^{\pm}$ *denotes the moduli space with respect to* $(f^{\pm}, g^{\pm})$.

Exercise 1.35. Prove (1.3) using Theorem 1.34.

Next, we have the following *functoriality* result:

Theorem 1.36 (*Functoriality in Morse theory*)**.** *Suppose* (f^-, g^-), (f^0, g^0) *and* (f^+, g^+) *are three Morse-Smale pairs, which give rise to three maps*

$$\phi_-^0 : \mathrm{HM}_*(f^-, g^-) \to \mathrm{HM}_*(f^0, g^0),$$

$$\phi_0^+ : \mathrm{HM}_*(f^0, g^0) \to \mathrm{HM}_*(f^+, g^+),$$

$$\phi_-^+ : \mathrm{HM}_*(f^-, g^-) \to \mathrm{HM}_*(f^+, g^+).$$

Then one has

$$\phi_-^+ = \phi_0^+ \circ \phi_-^0.$$

Corollary 1.37. *The map* $\phi_-^+ : \mathrm{HM}_*(f^-, g^-) \to \mathrm{HM}_*(f^+, g^+)$ *is an isomorphism with inverse* $\phi_+^- : \mathrm{HM}_*(f^+, g^+) \to \mathrm{HM}_*(f^-, g^-)$*. Thus the Morse homology* $\mathrm{HM}_*(f, g)$ *is independent of the Morse-Smale pair* (f, g)*.*

Exercise 1.38. Prove Corollary 1.37.

Corollary 1.37 implies that the following definition makes sense.

Definition 1.39. We define the *Morse homology* $\mathrm{HM}_*(B)$ of a closed manifold B to be the Morse homology $\mathrm{HM}_*(f, g)$ for some (and hence any) Morse-Smale pair (f, g) on B.

In fact, the Morse homology is the same as the singular homology.

Theorem 1.40 (*The Morse homology theorem*)**.** *There is a canonical isomorphism*

$$\mathrm{HM}_*(B) \cong \mathrm{H}_*(B; \mathbb{Z}_2).$$

There are many ways to prove Theorem 1.40. One way is show that a Morse-Smale pair induces a *cellular filtration* of the manifold B. Nevertheless, this argument is of a somewhat different flavour, and hence we will not discuss it in the course.

Exercise 1.41. Read and study the excellent set [AM06] of lecture notes, focusing in particular on the proof of the Morse homology theorem.

We now discuss an extension of the theory presented above to the *Morse-Bott* setting. In the setup we present here, this theory is due to Frauenfelder [Fra04, Appendix A]. As with the previous section, we will not prove the assertions we make here (in this course we will only prove *Floer-theoretic* results!). However, proofs of all the assertions made can be found in [Fra04, Appendix A]. An alternative approach can be found in [BH13].

We begin by defining a *Morse-Bott* function. As before, we assume for simplicity that B is a closed finite dimensional manifold.

Definition 1.42. Suppose $f \in C^\infty(B)$. We say that f is a *Morse-Bott function* if $\mathrm{crit}(f)$ is a closed submanifold of B (whose components may be of differing dimensions), which moreover has the property that if $x \in \mathrm{crit}(f)$ and $C_x \subset B$ denotes the connected component of $\mathrm{crit}(f)$ containing x then

$$\ker\left(D^v(df)(x) : T_xB \to T_x^*B\right) = T_xC_x,$$

where $D^v(df)(x)$ denotes the Hessian of f at x, as in Definition 1.10.

Remark 1.43. Thus a Morse function is simply a Morse-Bott function with the additional property that $C_x = \{x\}$ for each $x \in \mathrm{crit}(f)$.

Compactness of B implies that if f is a Morse-Bott function on B then $\mathrm{crit}(f)$ has finitely many components

$$\mathrm{crit}(f) = \bigsqcup_{j=1}^{N} C_j. \tag{1.5}$$

In order to set up the Morse-Bott complex we start by picking a collection

$$h_j : C_j \to \mathbb{R}$$

of Morse functions on each component C_j. It will sometimes be convenient to regard the collection $\{h_j\}$ as a single Morse function

$$h : \mathrm{crit}(f) \to \mathbb{R};$$

this is harmless as the C_j are all pairwise disjoint. One should beware though that if we think of h as a single function on $\mathrm{crit}(f)$, then h is defined on a space whose components may be of differing dimensions.

Note that if $x \in \mathrm{crit}(f)$ the Morse index $\mathrm{ind}_f(x)$ still makes sense (it is still the number of strictly negative eigenvalue of the linear map $D^v(df)(x)$). However if $x \in \mathrm{crit}(h_j)$ then we now have two different Morse indices: the index $\mathrm{ind}_f(x)$ of x as a critical point of f, and the index $\mathrm{ind}_{h_j}(x)$ of x as a critical point of h_j. It turns out the correct thing to do is to sum the two indices:

Definition 1.44. Given $x \in \mathrm{crit}(h) \subset \mathrm{crit}(f)$, we define the *Morse-Bott index* of x, written

$$\mathrm{ind}_{f,h}(x) := \mathrm{ind}_f(x) + \mathrm{ind}_{h_j}(x),$$

where $x \in C_j$. We define

$$\mathrm{crit}_k(f,h) := \{x \in \mathrm{crit}(h) \mid \mathrm{ind}_{f,h}(x) = k\}.$$

We can now define the Morse-Bott chain complex:

Definition 1.45. Given a Morse-Bott function $f \in C^\infty(B)$, a Morse function $h \in C^\infty(\mathrm{crit}(f))$, and $k \geq 0$, we define the Morse-Bott chain complex by

$$\mathrm{CMB}_k(f,h) := \bigoplus_{x \in \mathrm{crit}_k(f,h)} \mathbb{Z}_2 \langle x \rangle .$$

In order to define the boundary operator $\partial : \mathrm{CMB}_k(f,h) \to \mathrm{CMB}_{k-1}(f,h)$ we need to introduce *gradient flow lines with cascades*. This is where things start to get messy...

Fix a Riemannian metric g on B, and fix Riemannian metrics ρ_j on C_j for $j = 1, \dots, N$. As with h, it is convenient to regard the $\{\rho_j\}$ as defining a single metric ρ on $\mathrm{crit}(f)$. We will use g to define the gradient vector field $\nabla_g f$, and use ρ to define the gradient vector field $\nabla_\rho h$. In what follows given $x \in \mathrm{crit}(h_j)$ we denote by $W^u(x; -\nabla_\rho h)$ the *unstable manifold* of x with respect to the flow $\phi^s : C_j \to C_j$ of $-\nabla_{\rho_j} h_j$:

$$W^u(x; -\nabla_\rho h) := \left\{ y \in C_j \mid \lim_{s \to -\infty} \phi^s(y) = x \right\}.$$

Similarly the *stable manifold* $W^s(x; -\nabla_\rho h)$ is the set

$$W^s(x; -\nabla_\rho h) := \left\{ y \in C_j \mid \lim_{s \to +\infty} \phi^s(y) = x \right\}.$$

Definition 1.46. Fix $1 \le l, m \le N$ (where N was defined in (1.5)), and fix critical points $x \in \mathrm{crit}(h_l)$ and $y \in \mathrm{crit}(h_m)$, and $k \ge 1$. An element of $\mathcal{M}_k^c(x,y)$ is a tuple $(2k-1)$-tuple

$$(\mathbf{u} = (u_1, \dots, u_k), \mathbf{t} = (t_1, \dots, t_{k-1})),$$

where $u_j \in C^\infty(\mathbb{R}, B)$ and $t_j \ge 0$ are such that:

1. Each u_j is a non-constant gradient flow line of f:

$$\partial_s u_j + \nabla_g f(u_j) = 0.$$

2. The first flow line u_1 satisfies

$$\lim_{s \to -\infty} u_1(s) \in W^u(x; -\nabla_\rho h),$$

   and the last flow line u_k satisfies

$$\lim_{s \to +\infty} u_k(s) \in W^s(y; -\nabla_\rho h).$$

3. For $1 \le j \le k-1$ there are critical submanifolds C_{i_j} and gradient flow lines $v_j \in C^\infty(\mathbb{R}, C_{i_j})$ of h_{i_j}:

$$\partial_s v_j + \nabla_{\rho_{i_j}} h_{i_j}(v_j) = 0,$$

   such that

$$\lim_{s \to +\infty} u_j(s) = v_j(0),$$

$$\lim_{s \to -\infty} u_{j+1}(s) = v_j(t_j).$$

One should think of the v_j as 'cascades', as per the following picture: [insert picture]. The 'c' in $\mathcal{M}^c$ stands for 'cascades'.

There is a free $\mathbb{R}$-action on each flow line u_j, and hence $\mathcal{M}_k(x,y)$ admits a free $\mathbb{R}^k$-action. As before we denote by $\underline{\mathcal{M}}_k^c(x,y)$ the quotient space

$$\underline{\mathcal{M}}_k^c(x,y) := \mathcal{M}_k^c(x,y)/\mathbb{R}^k.$$

If $l = m$ then we let $\mathcal{M}_0(x,y)$ denote the set of normal gradient flow lines of h running from x to y, and as usual $\underline{\mathcal{M}}_0^c(x,y)$ is then the quotient space $\mathcal{M}_0^c(x,y)/\mathbb{R}$. If x and y do not belong to the same component then we set $\mathcal{M}_0(x,y) := \emptyset$. Finally we set

$$\mathcal{M}^c(x,y) := \bigcup_{k=0}^{\infty} \mathcal{M}_k^c(x,y),$$

and

$$\underline{\mathcal{M}}^c(x,y) := \bigcup_{k=0}^{\infty} \underline{\mathcal{M}}_k^c(x,y)$$

In other words, the space $\underline{\mathcal{M}}^c(x,y)$ is the space of *gradient flow lines with arbitrarily many cascades*.

Remark 1.47. If $l = m$ then $\underline{\mathcal{M}}_k^c(x,y) = \emptyset$ for all $k \ge 1$, whereas if $l \ne m$ then $\underline{\mathcal{M}}_0^c(x,y) = \emptyset$.

We now have the following theorem, which is due to Frauenfelder [Fra04, Appendix A]. This should be contrasted to Theorem 1.15, and is proved in essentially the same way.

Theorem 1.48. *The space* $\underline{\mathcal{M}}^c(x,y)$ *has virtual dimension* $\mathrm{ind}_{f,h}(x) - \mathrm{ind}_{f,h}(y) - 1$.

This is complemented by the following result, which is proved in the same way as Theorem 1.18. This result is again due to Frauenfelder.

Theorem 1.49. *Suppose* $f \in C^\infty(B)$ *is a Morse-Bott function, and* $h : \mathrm{crit}(f) \to \mathbb{R}$ *is a Morse function, and* ρ *is a Riemannian metric on* $\mathrm{crit}(f)$ *such that* (h,ρ) *is a Morse-Smale pair. Then there is a set* $\mathcal{R}_{f,h,\rho} \subset \mathsf{Met}(B)$ *of second category with the property that if* $g \in \mathcal{R}_{f,h,\rho}$ *then the moduli spaces* $\mathcal{M}^c(x,y)$ *are all cut out transversely, and hence are manifolds.*

Finally one has the following analogue of Theorem 1.20, which again is due to Frauenfelder.

Theorem 1.50. *Suppose* $\mathrm{ind}_{f,h}(x) = \mathrm{ind}_{f,h}(y) + 1$*. Then the space* $\underline{\mathcal{M}}^c(x,y)$ *is compact, and hence a finite set.*

This means that one can define numbers $n^c(x,y)$ for $x,y \in \mathrm{crit}(h)$ by

$$n^c(x,y) := \#_2 \underline{\mathcal{M}}^c(x,y).$$

Then as in Definition 1.23 one defines the boundary operator

$$\partial : \mathrm{CMB}_k(f,h) \to \mathrm{CMB}_{k-1}(f,h)$$

by setting

$$\partial \langle x \rangle := \sum_{y \in \mathrm{crit}_{k-1}(f,h)} n^c(x,y) \langle y \rangle ,$$

and then extending by linearity. The proof that $\partial^2 = 0$ is similar to that of Corollary 1.31, and involves studying the boundary $\partial \overline{\underline{\mathcal{M}}^c(x,y)}$ of the compactification of the space $\underline{\mathcal{M}}^c(x,y)$ when $\mathrm{ind}_{f,h}(x) = \mathrm{ind}_{f,h}(y) + 2$.

Definition 1.51. A *Morse-Bott-Smale quadruple* is a quadruple (f,h,g,ρ) consisting of a Morse-Bott function $f \in C^\infty(B)$, a Morse function $h \in C^\infty(\mathrm{crit}(f))$, a Riemannian metric ρ on $\mathrm{crit}(f)$ such that (h,ρ) is a Morse-Smale pair, and finally a Riemannian metric $g \in \mathcal{R}_{f,h,\rho}$.

The upshot of our work so far is that we can now speak of the *Morse-Bott homology* $\mathrm{HMB}_*(f,h,g,\rho)$ of a Morse-Bott-Smale quadruple.

Finally let us discuss independence. Suppose (f^-,h^-,g^-,ρ^-) and (f^+,h^+,g^+,ρ^+) are two Morse-Bott-Smale quadruples. As in (1.1) we wish to define a chain map

$$\Phi_-^+ : \mathrm{CMB}_k(f^-,h^-,g^-,\rho^-) \to \mathrm{CMB}_k(f^+,h^+,g^+,\rho^+).$$

This will be done by defining a space $\mathcal{N}^c(x^-,x^+)$ and setting

$$\Phi_-^+ \langle x^- \rangle := \sum_{x^+ \in \mathrm{crit}_k(f^+,h^+)} n_-^{c+}(x^-,x^+) \langle x^+ \rangle ,$$

where $n_-^{c+}(x^-,x^+) := \#_2 \mathcal{N}^c(x^-,x^+)$. In order to define $\mathcal{N}^c(x^-,x^+)$, as before we begin by choosing a homotopy $(f_s,g_s)_{s\in\mathbb{R}}$ such that

$$(f_s,g_s) = \begin{cases} (f^-,g^-), & s \le -T, \\ (f^+,g^+), & s \ge T, \end{cases}$$

for some $T > 0$. Fix $x^- \in \operatorname{crit}(h^-)$ and $x^+ \in \operatorname{crit}(h^+)$. The space $\mathcal{N}^c(x^-, x^+)$ is defined as the union

$$\mathcal{N}^c(x^-, x^+) = \bigcup_{k=0}^{\infty} \bigcup_{l=1}^{k} \mathcal{N}^c_{k,l}(x^-, x^+),$$

where an element of $\mathcal{N}^c_{k,l}(x^-, x^+)$ is a tuple $(2k-1)$-tuple

$$(\mathbf{u} = (u_1, \ldots, u_k), \mathbf{t} = (t_1, \ldots, t_{k-1}))\,,$$

where $u_j \in C^\infty(\mathbb{R}, B)$ and $t_j \geq 0$ are such that:

1. For $1 \leq j \leq l$, each u_j is a non-constant gradient flow line of f^-:

$$\partial_s u_j + \nabla_{g^-} f^-(u_j) = 0.$$

   For $l+1 \leq j \leq k$, each u_j is a non-constant gradient flow line of f^+:

$$\partial_s u_j + \nabla_{g^+} f^+(u_j) = 0.$$

   For $j = l$ the map u_l is a (possibly constant) flow line of the s-dependent equation

$$\partial_s u_l + \nabla_{g_s} f_s(u_l) = 0.$$

2. The first flow line u_1 satisfies

$$\lim_{s \to -\infty} u_1(s) \in W^u(x^-; -\nabla_{\rho^-} h^-),$$

   and the last flow line u_k satisfies

$$\lim_{s \to +\infty} u_k(s) \in W^s(x^+; -\nabla_{\rho^+} h^+).$$

3. For $1 \leq j \leq l-1$ there are critical submanifolds C_{i_j} and gradient flow lines $v_j \in C^\infty(\mathbb{R}, C_{i_j})$ of $h^-_{i_j}$:

$$\partial_s v_j + \nabla_{\rho^-_{i_j}} h^-_{i_j}(v_j) = 0$$

   such that

$$\lim_{s \to +\infty} u_j(s) = v_j(0),$$
$$\lim_{s \to -\infty} u_{j+1}(s) = v_j(t_j).$$

   For $l \leq j \leq k-1$ there there are critical submanifolds C_{i_j} and gradient flow lines $v_j \in C^\infty(\mathbb{R}, C_{i_j})$ of $h^+_{i_j}$:

$$\partial_s v_j + \nabla_{\rho^+_{i_j}} h^+_{i_j}(v_j) = 0$$

   such that

$$\lim_{s \to +\infty} u_j(s) = v_j(0),$$
$$\lim_{s \to -\infty} u_{j+1}(s) = v_j(t_j).$$

Just like in (1.4), one can prove that the map Φ^+_- is a chain map, and hence induces a map

$$\phi^+_- : \mathrm{HMB}_*(f^-, h^-, g^-, \rho^-) \to \mathrm{HMB}_*(f^+, h^+, g^+, \rho^+).$$

This is again done by studying the boundary components $\partial\overline{\mathcal{N}^c(x^-, x^+)}$ when $\operatorname{ind}_{f^-,h^-}(x^-) = \operatorname{ind}_{f^+,h^+}(x^+) + 1$. Finally we again have the following *functoriality result*:

Theorem 1.52 (*Functoriality in Morse-Bott theory*)**.** *Suppose* (f^-, h^-, g^-, ρ^-), (f^0, h^0, g^0, ρ^0) *and* (f^+, h^+, g^+, ρ^+) *are three Morse-Bott-Smale quadruples, which give rise to three maps*

$$\phi_-^0 : \mathrm{HMB}_*(f^-, h^-, g^-, \rho^-) \to \mathrm{HMB}_*(f^0, h^0, g^0, \rho^0),$$
$$\phi_0^+ : \mathrm{HMB}_*(f^0, h^0, g^0, \rho^0) \to \mathrm{HMB}_*(f^+, h^+, g^+, \rho^+),$$
$$\phi_-^+ : \mathrm{HMB}_*(f^-, h^-, g^-, \rho^-) \to \mathrm{HMB}_*(f^+, h^+, g^+, \rho^+).$$

Then one has

$$\phi_-^+ = \phi_0^+ \circ \phi_-^0.$$

As in Corollary 1.37, this tells us that:

Corollary 1.53. *The Morse-Bott homology* $\mathrm{HMB}_*(f, h, g, \rho)$ *is independent of the choice of Morse-Bott-Smale quadruple.*

Since a Morse function is a particular case of a Morse-Bott function (cf. Remark 1.43), a Morse-Smale pair (f, g) gives rise to a Morse-Bott-Smale quadruple $(f, 0, g, 0)$. But we already know that the Morse homology is independent of the Morse-Smale pair. This proves:

Theorem 1.54 (*The Morse-Bott homology theorem*)**.** *If* (f, h, g, ρ) *is any Morse-Bott-Smale quadruple, then one has*

$$\mathrm{HMB}_*(f, h, g, \rho) \cong \mathrm{HM}_*(B) \cong \mathrm{H}_*(B; \mathbb{Z}_2).$$

Exercise 1.55. Fill in the details of Theorem 1.54.

CHAPTER 2

The Arnold Conjecture

Let (Q, ω) denote a closed *symplectic* manifold of dimension $2n$. By definition this means that $\omega^n \in \Omega^{2n}(Q)$ is a volume form. Equivalently, ω defines a bijective map

$$\Phi_\omega : \Omega^1(Q) \to \mathrm{Vect}(Q)$$

by associating to a 1-form ζ the vector field X_ζ defined by

$$\omega(X_\zeta, \cdot) = \zeta(\cdot).$$

For historical reasons one calls a smooth function $H \in C^\infty(Q)$ a *Hamiltonian*.

Definition 2.1. The *Hamiltonian vector field* X_H associated to $H \in C^\infty(Q)$ is by definition $X_H := \Phi_\omega(-dH)$. If $H \in C^\infty(S^1 \times Q)$, written either as $H(t, q)$ or $H_t(q)$, then we obtain a time dependent vector field $X_H(t, \cdot) = X_{H_t}(\cdot)$ where $X_{H_t} = \Phi_\omega(-dH_t)$. We denote by φ_H^t the flow of X_H and call the time-1 map $\varphi_H := \varphi_H^1$ a *Hamiltonian diffeomorphism*. The set of Hamiltonian diffeomorphisms forms a group $\mathrm{Ham}(Q, \omega)$.

Suppose $\varphi \in \mathrm{Ham}(Q, \omega)$. Then by definition there exists $H \in C^\infty(S^1 \times Q)$ such that $\varphi = \varphi_H^1$. In this case we say that H *generates* φ and write $H \mapsto \varphi$. The function H is certainly not unique: suppose $G \in C^\infty(S^1 \times Q)$ satisfies $\varphi_G^1 = \mathbb{1}$. Then if

$$(G\#H)(t, q) := G(t, q) + H(t, (\varphi_G^t)^{-1}(q)) \tag{2.1}$$

then one can readily check that

$$\varphi_{G\#H}^t = \varphi_G^t \circ \varphi_H^t.$$

In particular, in this case one has $G\#H \mapsto \varphi$ as well.

Exercise 2.2. Suppose $\varphi \in \mathrm{Ham}(Q, \omega)$. Show that one can choose $H \in C^\infty(S^1 \times Q)$ such that $H \mapsto \varphi$ and such that there exists $\varepsilon > 0$ such that $H(t, \cdot) \equiv 0$ whenever $t \in (-\varepsilon, \varepsilon)$.

If $H, K \in C^\infty(S^1 \times Q)$ then we can define $H\#K$ in exactly the same way (2.1). However in general $H\#K$ may not be 1-periodic. Nevertheless $H\#K$ *is* 1-periodic if either (a) $\varphi_H^1 = \mathbb{1}$ or (b) K satisfies the conditions imposed in Exercise 2.2.

Remark 2.3. Since the object of interest is always the Hamiltonian diffeomorphism φ rather than the particular function H, and since Exercise 2.2 implies that we may always choose H such that $H(t, \cdot) \equiv 0$ whenever t is sufficiently close to 0, from now on we will always implicitly assume whenever convenient that this is the case. Thus by convention $H\#K$ is always 1-periodic.

Exercise 2.4. Show that if $H \mapsto \varphi$ then $\overline{H} \mapsto \varphi^{-1}$, where

$$\overline{H}(t, q) := -H(t, \varphi_H^t(q)).$$

Conclude that $\mathrm{Ham}(Q, \omega)$ really is a group.

Remark 2.5. Suppose $t \mapsto \varphi_t : t \in [0, 1]$ is a path in $\mathrm{Ham}(Q, \omega)$ with $\varphi_0 = \mathbb{1}$. Then it is in fact always possible to choose a Hamiltonian $H \in C^\infty(S^1 \times Q)$ such that $\varphi_t = \varphi_H^t$.

Exercise 2.6. Convince yourself that Remark 2.5 is *not* obvious. Then consult [MS98, Proposition 10.17] and study the proof.

We denote by $\mathrm{Symp}_0(Q,\omega)$ the set of diffeomorphisms $\psi : Q \to Q$ that satisfy $\psi^*\omega = \omega$ and that are isotopic to the identity through such maps.

Lemma 2.7. *One has* $\mathrm{Ham}(Q,\omega) \subset \mathrm{Symp}_0(Q,\omega)$.

Proof. Suppose $\varphi \in \mathrm{Ham}(Q,\omega)$, and let $H \mapsto \varphi$. Compute:

$$\begin{aligned} \frac{\partial}{\partial t}(\varphi_H^t)^*\omega &= L_{X_{H_t}}\omega \\ &= (di_{X_{H_t}} + i_{X_{H_t}}d)\omega \\ &= d(-dH_t)) \\ &= 0. \end{aligned}$$

Since $\varphi_H^0 = \mathbb{1}$ we have $(\varphi_H^0)^*\omega = \omega$ and hence also $(\varphi_H^1)^*\omega = \omega$. Thus $\mathrm{Ham}(Q,\omega) \subset \mathrm{Symp}(Q,\omega)$. Since $\mathrm{Ham}(Q,\omega)$ is path-connected, the result follows. ■

Exercise 2.8. Suppose $\psi \in \mathrm{Symp}_0(Q,\omega)$ and $\varphi \in \mathrm{Ham}(Q.\omega)$. Show that if $H \mapsto \varphi$ then $H \circ \psi \mapsto \psi \circ \varphi \circ \psi^{-1}$. Thus $\mathrm{Ham}(Q,\omega)$ is a *normal* subgroup of $\mathrm{Symp}_0(Q,\omega)$.

Remark 2.9. A celebrated of Banyaga [Ban78] says that $\mathrm{Ham}(Q,\omega)$ is a *simple group*.

Remark 2.10. Suppose $t \mapsto \varphi_t : t \in S^1$ is a *loop* of Hamiltonian diffeomorphisms such that $\varphi_0 = \mathbb{1}$. Fix $q \in Q$ and consider the loop $x_q(t) := \varphi_t(q)$. One can ask: is the class $[x_q] \in \pi_1(Q,q)$ zero? That is, is x_q a contractible loop? This would imply that the evaluation map

$$\pi_1(\mathrm{Ham}(Q,\omega)) \to \pi_1(Q)$$

is always trivial. Amazingly, the answer is yes, but to prove this one needs the entire machinery of Floer homology. Here is a sketch of the proof: choose t_0 very close to 1 so that the paths $t \mapsto \varphi_t(q) : t \in [t_0, 1]$ are all contained in geodesically convex subsets of Q. Floer homology guarantees the existence of at least *one* point $q_0 \in Q$ such that $\varphi_{t_0}(q_0) = q_0$ and such that the loop $t \mapsto \varphi_t(q_0) : t \in [0, t_0]$ is contractible (see Remark 2.27 below). Since $\varphi_t(q_0) : t \in [t_0, 1]$ is contained in a geodesically convex set, the loop $t \mapsto \varphi_t(q_0) : t \in [t_0, 1]$ is also contractible. Thus the concatenated loop $t \mapsto \varphi_t(q_0) : t \in [0, 1]$ is contractible. Now consider the map

$$\mathsf{ev}_\varphi : Q \to C^\infty(S^1, Q)$$

given by

$$\mathsf{ev}_\varphi(q) := x_q.$$

Since Q is connected (we always assume this!), the map ev has its image in one connected component of $C^\infty(S^1, Q)$. Since $\mathsf{ev}_\varphi(q_0)$ belongs to the connected component of $C^\infty(S^1, Q)$ containing the contractible loops, this implies that $\mathsf{ev}_\varphi(q)$ belongs to the connected component of $C^\infty(S^1, Q)$ containing the contractible loops for every $q \in Q$. Thus $[x_q] = 0 \in \pi_1(Q,q)$ for each $q \in Q$.

Definition 2.11. Suppose $\varphi \in \mathrm{Ham}(Q,\omega)$ and $q \in \mathrm{fix}(\varphi)$. Then if $H \mapsto \varphi$ then the path $x(t) := \varphi_H^t(q)$ satisfies

$$x'(t) = X_H(t,x).$$

We call x a *1-periodic orbit* of X_H and write $x \in \mathcal{P}_1(H)$. Thus there is a bijection

$$\mathcal{P}_1(H) \to \mathrm{fix}(\varphi_H)$$

given by $x(t) \mapsto x(0)$. Denote by $\mathcal{P}_1^\circ(H) \subset \mathcal{P}_1(H)$ the set of 1-periodic orbits $x(t)$ that are contractible.

Exercise 2.12. Suppose $\varphi \in \mathrm{Ham}(Q, \omega)$. Define $\mathrm{fix}^\circ(\varphi) \subset \mathrm{fix}(\varphi)$ by saying

$$\mathrm{fix}^\circ(\varphi) := \{x(0) \mid x \in \mathcal{P}_1^\circ(H), \text{ for some } H \mapsto \varphi\}.$$

Prove that $\mathrm{fix}^\circ(\varphi)$ is well defined (i.e. independent of the choice of $H \mapsto \varphi$). *Hint:* Use Remark 2.10.

Definition 2.13. We say that $x \in \mathcal{P}_1^\circ(H)$ is *non-degenerate* if the linear map

$$D\varphi_H^1(x(0)) : T_{x(0)}Q \to T_{x(1)}Q$$

does not have 1 as an eigenvalue. We say that H is *non-degenerate* if every element of $\mathcal{P}_1^\circ(H)$ is non-degenerate. Note that non-degeneracy is clearly a property of the time-1 map φ_H^1 (i.e. if $H \mapsto \varphi$ and $K \mapsto \varphi$ then H is non-degenerate if and only if K is). Thus let us say that $\varphi \in \mathrm{Ham}(Q, \omega)$ is *non-degenerate* if there exists $H \mapsto \varphi$ such that H is non-degenerate.

Remark 2.14. As we will in Lemma 3.20, saying that $x \in \mathcal{P}_1^\circ(H)$ is non-degenerate is equivalent to saying that x is a regular zero of a certain section σ_H of a Banach bundle in the sense of Definition 1.3.

Non-degeneracy is a generic property in the following sense: given $H \in C^\infty(S^1 \times Q)$ and $\varepsilon > 0$ there exists $\tilde{H} \in C^\infty(S^1 \times Q)$ such that $\left\| H - \tilde{H} \right\|_{C^\infty(S^1 \times Q)} < \varepsilon$ and such that $\tilde{H}$ is non-degenerate.

Exercise 2.15. Do not be fooled into thinking that all 'reasonable' Hamiltonians are non-degenerate. Indeed, suppose $H \in C^\infty(Q)$ is *autonomous* (i.e. $H = H(q)$ does not depend on $t \in S^1$). Prove that every $x \in \mathcal{P}_1(H)$ is always degenerate!

Let us now state a weak form of the celebrated *Arnold Conjecture*. The main aim of this course is to prove this conjecture for a certain special class of symplectic manifolds.

Conjecture 2.16. *(The Arnold Conjecture)*
Suppose $H \in C^\infty(S^1 \times Q)$ is non-degenerate. Then

$$\#\mathcal{P}_1^\circ(H) \geq \sum_{k=0}^{2n} \dim \mathrm{H}_k(Q; \mathbb{Z}_2).$$

Remark 2.17. It follows readily from the Morse homology Theorem 1.40 that if $f \in C^\infty(B)$ is a Morse function then

$$\#\mathrm{crit}(f) \geq \sum_{k=0}^{\dim B} \dim \mathrm{H}_k(B; \mathbb{Z}_2).$$

The Arnold Conjecture as stated is a statement about $\mathcal{P}_1^\circ(H)$. Using Exercise 2.12, the Arnold Conjecture can actually be interpreted as a statement about the set $\mathrm{fix}^\circ(\varphi)$ for non-degenerate $\varphi \in \mathrm{Ham}(Q, \omega)$. Namely, the Arnold Conjecture asserts that if $\varphi \in \mathrm{Ham}(Q, \omega)$ is non-degenerate then $\#\mathrm{fix}^\circ(\varphi)$ is at least as big as the sum of the Betti numbers of Q (with $\mathbb{Z}_2$-coefficients). However since Remark 2.10 used the existence of Floer homology, for now we don't actually know that the set $\mathrm{fix}^\circ(\varphi)$ is even well defined! So for now we will content ourselves with stating the Arnold Conjecture for non-degenerate Hamiltonians $H \in C^\infty(S^1 \times Q)$ only (one could alternatively state the weaker conjecture that the set $\#\mathrm{fix}(\varphi)$ of *all* fixed points is at least as big as the sum of the Betti numbers of Q whenever $\varphi \in \mathrm{Ham}(Q, \omega)$ is non-degenerate).

Let us now define the class of symplectic manifolds which we will work with in this course.

Definition 2.18. Let (Q, ω) denote a symplectic manifold. There is a well defined map

$$I_\omega : \pi_2(Q) \to \mathbb{R}$$

given by

$$I_\omega([v]) := \int_{S^2} v^* \omega.$$

There is a second homomorphism $I_{c_1} : \pi_2(Q) \to \mathbb{Z}$ we can associate to (Q, ω). This will require a little bit more work to define.

Definition 2.19. We will use the sign convention that an almost complex structure J on Q (that is, a section $J \in \Gamma(Q, \mathrm{End}(TQ))$ such that $J^2 = -\mathbb{1}$) is *compatible* with the symplectic form if

$$g_J(\cdot, \cdot) := \omega(J\cdot, \cdot) \tag{2.2}$$

defines a Riemannian metric on Q.

Remark 2.20. The sign convention (2.2) is opposite to the one that you will find in many textbooks (e.g.. [MS98, MS12]), and in most papers. Throughout the subject there are many mutually inconsistent sign conventions in common use. In fact, it is not possible simultaneously have all the 'standard' sign conventions. In general everyone tries to use as many standard conventions as possible, and thus everyone has to pay the price somewhere. We will adopt the convention introduced by Abbondandolo and Schwarz [AS10] (which also happens to be the one I like and always use in my papers!). *If nothing else, the point of this course is to convince as many people as possible to adopt these sign conventions!*

It is well known that the set $\mathcal{J}(Q, \omega)$ of almost complex structures J on Q that are compatible with ω is path connected. Let us now fix one: then $(TQ, J) \to Q$ is a complex vector bundle, and hence gives rise to a cohomology class $c_1(TQ, J) \in \mathrm{H}^2(Q; \mathbb{Z})$.

Exercise 2.21. Prove that if $\tilde{J}$ was another element of $\mathcal{J}(Q, \omega)$ then

$$c_1(TQ, J) = c_1(TQ, \tilde{J}).$$

Thus we may unambiguously define

$$c_1(Q) := c_1(TQ, J) \quad \text{for any } J \in \mathcal{J}(Q, \omega).$$

Definition 2.22. Define

$$I_{c_1} : \pi_2(Q) \to \mathbb{Z}$$

by

$$I_{c_1}([v]) := \left\langle c_1(Q), v_*([S^2]) \right\rangle,$$

where $v_* : \mathrm{H}_2(S^2; \mathbb{Z}) \to \mathrm{H}_2(Q; \mathbb{Z})$ is the induced map in homology and $[S^2] \in \mathrm{H}_2(S^2; \mathbb{Z})$ is the fundamental class.

Exercise 2.23. Prove that both I_ω and I_{c_1} are well defined.

Definition 2.24. We say that (Q, ω) is *symplectically aspherical* if both I_ω and I_{c_1} are identically zero:

$$I_\omega \equiv I_{c_1} \equiv 0.$$

In this course we shall construct the Floer homology of a symplectically aspherical symplectic manifold. More precisely, we will prove:

Theorem 2.25. *Let (Q, ω) be a closed symplectically aspherical symplectic manifold. Let $H \in C^\infty(S^1 \times Q)$ be non-degenerate. Then there is a chain complex $(\mathrm{CF}_*(H), \partial)$ associated to H, which is generated by the elements of $\mathcal{P}_1^\circ(H)$:*

$$\mathrm{CF}_*(H) := \bigoplus_{x \in \mathcal{P}_1^\circ(H)} \mathbb{Z}_2 \langle x \rangle .$$

The homology of this complex is called the Floer homology *of H and is written*

$$\mathrm{H}_*(\mathrm{CF}_*(H), \partial) := \mathrm{HF}_*(H).$$

Moreover the Floer homology is canonically independent of the choice of H, and hence we can define the Floer homology *of (Q, ω) to be*

$$\mathrm{HF}_*(Q, \omega) := \mathrm{HF}_*(H) \quad \textit{for any non-degenerate } H \in C^\infty(S^1 \times Q).$$

Finally, there is a canonical isomorphism

$$\mathrm{HF}_*(Q, \omega) \cong \mathrm{H}_{*+n}(Q; \mathbb{Z}_2).$$

Exercise 2.26. Use Theorem 2.25 to prove the Arnold Conjecture 2.16 for symplectically aspherical manifolds.

Remark 2.27. There is also a 'degenerate' version of the Arnold Conjecture. Before stating this, recall that if $f \in C^\infty(B)$ is any (not necessarily Morse) function then *Ljusternik-Schnirelman* theory implies that

$$\#\mathrm{crit}(f) \geq \mathrm{cuplength}_{\mathbb{Z}_2}(B),$$

where $\mathrm{cuplength}_{\mathbb{Z}_2}(B)$ is the largest number N such that there exist cohomology classes $a_j \in \mathrm{H}^{\geq 1}(B; \mathbb{Z}_2)$ such that $a_1 \smile a_2 \cdots \smile a_N \neq 0$. See for instance [MS98, Theorem 11.16]. The degenerate form of the Arnold Conjecture states that if φ is *any* Hamiltonian diffeomorphism then

$$\#\mathrm{fix}^\circ(\varphi) \geq \mathrm{cuplength}_{\mathbb{Z}_2}(Q).$$

For symplectically aspherical manifolds this was proved by Rudyak and Oprea [RO99].

Remark 2.28. Finally we remark that a more ambitious conjecture would be to assert that for *any* symplectic manifold (Q, ω) and and ring R of coefficients, one has

$$\#\mathrm{fix}^\circ(\varphi) \geq \sum_{k=0}^{\dim Q} \dim \mathrm{H}_k(Q; R) \quad \text{for all non-degenerate } \varphi \in \mathrm{Ham}(Q, \omega),$$

and

$$\#\mathrm{fix}^\circ(\varphi) \geq \mathrm{cuplength}_R(Q) \quad \text{for all } \varphi \in \mathrm{Ham}(Q, \omega).$$

This is still open in many cases, although it has been proved in far more generality than just the closed symplectically aspherical case we study here.

Here is a related conjecture, which is due to Conley, which, as stated below, was recently proved by Ginzburg [Gin10]. Recall a *periodic point* of a diffeomorphism φ is a fixed point of some iterate φ^k. A periodic point is called *simple* if it is not an iterate of another periodic point. Finally, for Hamiltonian diffeomorphisms it again makes sense to speak of 'contractible' periodic points: $\mathrm{per}^\circ(\varphi)$:

$$\mathrm{per}^\circ(\varphi) := \left\{ x(0) \mid x \in \mathcal{P}_1^\circ(H^{\# k}), \text{ for some } H \mapsto \varphi \right\}$$

(here $H^{\# k} = \underbrace{H \# \dots \# H}_{k \text{ times}}$ generates the flow φ^k).

Conjecture 2.29. *(The Conley Conjecture)*

Suppose (Q, ω) *is a closed symplectic aspherical manifold. Let* $\varphi \in \mathrm{Ham}(Q, \omega)$. *Then* φ *has infinitely many contractible simple periodic points.*

Time permitting, we will prove a weaker version of this conjecture at the end of the course, which is due to Salamon and Zehnder [SZ92]. Unlike the Arnold Conjecture 2.16, which is expected to be true for all symplectic manifolds (cf. Remark 2.28), the Conley Conjecture is certainly *not* true for arbitrary symplectic manifolds, as the following exercise shows.

Exercise 2.30. Show that Conley Conjecture is false on $(S^2, \omega = \text{area form})$. *Hint:* Consider an irrational rotation.

CHAPTER 3

The Hamiltonian action functional

Definition 3.1. Let $\Lambda Q := C^{\infty}_{\mathsf{contr}}(S^1, Q)$ denote the set of *contractible* smooth loops $x : S^1 \to Q$. Sometimes it will be convenient to work with the *completed loop space* $\mathcal{L}Q := W^{1,2}_{\mathsf{contr}}(S^1, Q)$. Thus $\mathcal{L}Q$ is the completion of ΛQ with respect to the $W^{1,2}$-Sobolev norm. Since dim $S^1 = 1$, elements of $\mathcal{L}Q$ are continuous. The reason we will sometimes prefer $\mathcal{L}Q$ is that $\mathcal{L}Q$ carries the structure of a Banach manifold, where ΛQ is only a Frechet manifold.

Exercise 3.2. Convince yourself you know how to define Sobolev spaces for functions that take values in a *manifold*.

Definition 3.3. If we fix a Riemannian metric $g = \langle \cdot, \cdot \rangle$ on Q we can define a Hilbert product $\langle\langle \cdot, \cdot \rangle\rangle_1$ on $\mathcal{L}Q$ as follows:

$$\langle\langle \xi, \zeta \rangle\rangle_1 := \int_{S^1} \langle \xi(t), \zeta(t) \rangle \, dt + \int_{S^1} \langle \nabla_t \xi(t), \nabla_t \zeta(t) \rangle \, dt,$$

where if $\xi, \zeta \in \Gamma(x^* TQ)$ then ∇_t denotes the covariant derivative (with respect to the Levi-Civita connection of g) along the curve x.

Theorem 3.4. *The space* $(\mathcal{L}Q, \langle\langle \cdot, \cdot \rangle\rangle_1)$ *is a complete Riemannian Banach manifold.*

Nevertheless, in some sense the key idea of Floer [Flo88] was to ignore Theorem 3.4! Instead of working on a nice Hilbert manifold, his masterstroke was to realise that one could instead work on something far less pleasant...

Definition 3.5. Fix a Riemannian metric $g = \langle \cdot, \cdot \rangle$ on Q and equip ΛQ (not $\mathcal{L}Q$) with the L^2-inner product:

$$\langle\langle \xi, \zeta \rangle\rangle_0 := \int_{S^1} \langle \xi(t), \zeta(t) \rangle \, dt.$$

This L^2-metric is not complete, and hence does not define the structure of a Hilbert space on the tangent spaces to ΛQ.

Remark 3.6. In Floer theory we will study the "negative gradient flow" of the Hamiltonian action functional $\mathbb{A}_H$ (cf. Definition 3.10) with respect to an L^2-metric of the form given in Definition 3.5. This is not a "flow" in the strict sense of the word, since, as we will see, the problem is not well-posed. That is, the corresponding "flow" Φ_s of $-\nabla \mathbb{A}_H$ is not a real flow: it is not necessarily true that for any given $x \in \Lambda Q$ there exists a solution $u : \mathbb{R} \to \Lambda Q$ of the initial value problem

$$\partial_s u = -\nabla \mathbb{A}_H(u), \quad u(0) = x.$$

Such a flow is called *unregularized* [Flo88]. One could obtain a real flow by working instead with the stronger metric $\langle\langle \cdot, \cdot \rangle\rangle_1$ from Definition 3.3. If one's goal was to do Morse theory with the functional $\mathbb{A}_H$ then this would make sense. However, Morse theory does not work with $\mathbb{A}_H$, since, as we will see, the relevant Morse indices are always infinite.

Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$.

Definition 3.7. Suppose $x \in \Lambda Q$. Define a *capping* of x to be a map $\overline{x} \in C^{\infty}(\mathbb{D}, Q)$ such that $\overline{x}(e^{2\pi i t}) = x(t)$ for all $t \in S^1$. Similarly if $x \in \mathcal{L}Q$ a *capping* of x is any map $\overline{x} \in C^0(\mathbb{D}^2, Q) \cap W^{1,2}(\mathbb{D}^2, Q)$ such that $\overline{x}(e^{2\pi i t}) = x(t)$ for all $t \in S^1$.

The following lemma is the reason why we assume that the map $I_\omega : \pi_2(Q) \to \mathbb{R}$ from Definition 2.18 is identically zero.

Lemma 3.8. *Suppose $x \in \Lambda Q$. Let $\overline{x}$ and $\overline{y}$ be two cappings of x. Then*

$$\int_{\mathbb{D}} \overline{x}^*\omega = \int_{\mathbb{D}} \overline{y}^*\omega.$$

Proof. Without loss of generality we may assume that there exists $\varepsilon > 0$ such that

$$\overline{x}(z) = \overline{x}(z/\,|z|) \quad \text{and} \quad \overline{y}(z) = \overline{y}(z/\,|z|)$$

for all $z \in \mathbb{D}$ with $1 - \varepsilon \leq |z| \leq 1$. Then there is a well defined map

$$v : S^2 = \mathbb{C} \cup \{\infty\} \to Q$$

defined by

$$v(z) = \begin{cases} \overline{x}(z), & |z| \leq 1, \\ \overline{y}(1/\,|z|), & |z| \geq 1. \end{cases}$$

Then

$$\int_{\mathbb{D}} \overline{x}^*\omega - \int_{\mathbb{D}} \overline{y}^*\omega = \int_{S^2} v^*\omega = I_\omega([v]) = 0.$$

∎

Exercise 3.9. Check that if $x \in \mathcal{L}Q$ then Lemma 3.8 still holds.

We now finally define the Hamiltonian action functional.

Definition 3.10. Let $H \in C^\infty(S^1 \times Q)$ and define the *Hamiltonian action functional*

$$\mathbb{A}_H : \Lambda Q \to \mathbb{R}$$

by

$$\mathbb{A}_H(x) := \int_{\mathbb{D}} \overline{x}^*\omega - \int_{S^1} H_t(x(t))dt,$$

where $\overline{x}$ is any capping of x. Note that $\mathbb{A}_H$ extends to a well defined functional on $\mathcal{L}Q$ by Exercise 3.9.

The following result is key to all that follows:

Lemma 3.11. *It holds that*

$$\operatorname{crit}(\mathbb{A}_H) = \mathcal{P}_1^\circ(H).$$

Proof. Fix $x \in \Lambda Q$ and let $\xi \in \Gamma(x^*TQ)$. Choose a path $(x_s)_{s\in(-\varepsilon,\varepsilon)} \subset \Lambda Q$ such that $x_0 = x$ and $\xi = \frac{\partial}{\partial s}\big|_{s=0} x_s$. Let $(\overline{x}_s) \subset C^\infty(\mathbb{D}, Q)$ denote a smooth family of cappings of x_s, and set define $\overline{\xi} \in \Gamma(\overline{x}^*TQ)$ by

$$\overline{\xi}(z) := \frac{\partial}{\partial s}\Big|_{s=0} \overline{x}_s(z).$$

Now we compute:

$$\begin{aligned} d\mathbb{A}_H(x)[\xi] &= \frac{\partial}{\partial s}\Big|_{s=0} \mathbb{A}_H(x_s) \\ &= \frac{\partial}{\partial s}\Big|_{s=0} \left(\int_{\mathbb{D}} \overline{x}_s^*\omega - \int_{S^1} H_t(x_s(t))dt \right). \end{aligned}$$

Now

$$\frac{\partial}{\partial s}\Big|_{s=0}\int_{\mathbb{D}} \overline{x}_s^*\omega = \int_{\mathbb{D}} \overline{x}_0^*(L_{\overline{\xi}}\omega)$$
$$= \int_{\mathbb{D}} \overline{x}_0^*(di_{\overline{\xi}}\omega)$$
$$\overset{(*)}{=} \int_{S^1} x^*(i_\xi\omega)$$
$$= \int_{S^1} \omega(\xi(t), x'(t))dt,$$

where $(*)$ used Stokes' theorem. Similarly

$$\frac{\partial}{\partial s}\Big|_{s=0}\int_{S^1} H_t(x_s(t))dt = \int_{S^1} dH_t(x(t))[\xi(t)]dt = -\int_{S^1} \omega(X_{H_t}(x(t)), \xi(t))dt.$$

Putting this together we see that

$$d\mathbb{A}_H(x)[\xi] = \int_{S^1} \omega(\xi(t), x'(t) - X_{H_t}(x(t))dt. \tag{3.1}$$

■

Definition 3.12. Suppose $\mathcal{J} \in \mathcal{J}(Q, \omega)$ is an ω-compatible almost complex structure (cf. Definition 2.19). Let $g_J := \omega(J\cdot, \cdot)$ denote the corresponding Riemannian metric, and let

$$\langle\langle \xi, \zeta \rangle\rangle_J := \int_{S^1} g_J(\xi(t), \zeta(t))dt$$

denote the L^2-inner product, as in Definition 3.5.

Definition 3.13. As above, suppose $J \in \mathcal{J}(Q, \omega)$ is an ω-compatible almost complex structure. Define a vector field $\nabla_J \mathbb{A}_H$on ΛQ by setting

$$\nabla_J \mathbb{A}_H(x) := J(x)(x'(t) - X_{H_t}(x(t))).$$

Lemma 3.14. *The vector field $\nabla_J \mathbb{A}_H$ is the L^2-gradient of $\mathbb{A}_H$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_J$. In other words,*

$$d\mathbb{A}_H(x)[\xi] = \langle\langle \nabla_J \mathbb{A}_H(x), \xi \rangle\rangle_J\,.$$

Proof. Immediate from (3.1):

$$d\mathbb{A}_H(x)[\xi] = \int_{S^1} \omega(\xi(t), x'(t) - X_{H_t}(x(t))dt$$
$$= \int_{S^1} \omega\left(J(x) \cdot J(x)(x'(t) - X_{H_t}(x(t))), \xi(t)\right) dt$$
$$= \int_{S^1} g_J(J(x)(x'(t) - X_{H_t}(x(t)), \xi(t))dt$$
$$= \langle\langle \nabla_J \mathbb{A}_H(x), \xi \rangle\rangle_J\,.$$

■

Now consider a Banach bundle $\mathcal{E} \to \mathcal{L}Q$ whose fibre over x is given by $L^2(S^1, x^*TQ)$. Define a section

$$\sigma_{J,H} : \mathcal{L}Q \to \mathcal{E} \tag{3.2}$$

by

$$\sigma_{J,H}(x) := J(x)(x'(t) - X_{H_t}(x(t))) = \nabla_J \mathbb{A}_H(x).$$

Lemma 3.15. *The vertical derivative $D^v\sigma_{J,H}(x)$ at $x \in \mathcal{P}_1^\circ(H)$ is given by*

$$D^v\sigma_{J,H}(x) : W^{1,2}(S^1, x^*TQ) \to L^2(S^1, x^*TQ),$$
$$D^v\sigma_{J,H}(x)[\xi] := J(x)(\nabla_t\xi - \nabla_\xi X_{H_t}(x)).$$

Proof. Choose a family $(x_s)_{s\in(-\varepsilon,\epsilon)} \subset \mathcal{L}Q$ such that $x_0 = x$ and $\frac{\partial}{\partial s}\big|_{s=0} x_s(t) = \xi(t)$. Then

$$\begin{aligned} D^v\sigma_{J,H}(x)[\xi] &= \frac{\partial}{\partial s}\Big|_{s=0}\sigma_{J,H}(x_s) \\ &= \frac{\partial}{\partial s}\Big|_{s=0} J(x_s)(x_s'(t) - X_{H_t}(x_s(t)) \\ &= \frac{\partial}{\partial s}\Big|_{s=0} J(x_s)\underbrace{\left(x'(t) - X_{H_t}(x(t))\right)}_{=0} + J(x)\frac{\partial}{\partial s}\Big|_{s=0}(x_s'(t) - X_{H_t}(x_s(t)) \\ &= J(x)\nabla_\xi x' - J(x)\nabla_\xi X_{H_t}(x) \end{aligned}$$

where the last line used the fact that the Levi-Civita connection is torsion free. ■

Let us now restrict to the linear setting for a while. We equip $\mathbb{R}^{2n}$ with the coordinates

$$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n),$$

and the standard Euclidean inner product $x \cdot y$. In order to minimize notational confusion we denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^{2n}$, so that given $z = (x, y)$ and $w = (u, v)$ we have

$$\langle z, w \rangle := x \cdot u + y \cdot v.$$

Define $\omega_0 \in \Omega^2(\mathbb{R}^{2n})$ by

$$\omega_0((x, y), (u, v)) := u \cdot y - x \cdot v.$$

Let $J_0 \in \mathbb{R}^{2n}$ be the matrix

$$J_0 := \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}, \tag{3.3}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix. Thus

$$\omega_0(J_0(x, y), (u, v)) = x \cdot u + y \cdot v = \langle (x, y), (u, v) \rangle .$$

The L^2 inner product on $L^2(S^1, \mathbb{R}^{2n})$ is given by

$$\langle\langle \xi, \zeta \rangle\rangle := \int_{S^1} \langle \xi, \zeta \rangle \, dt.$$

The following lemma is standard undergraduate functional analysis.

Lemma 3.16. *Let $S : S^1 \to \mathsf{Sym}(\mathbb{R}^{2n})$ denote a smooth loop of symmetric matrices. Let $\Psi : [0, 1] \to \mathsf{Sp}(\mathbb{R}^{2n}, \omega_0)$ denote the fundamental solution of the equation*

$$\Psi'(t) = J_0 \cdot S(t) \cdot \Psi(t), \quad \Psi(0) = \mathbb{1}.$$

Let

$$\sigma_S : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

denote the map

$$\sigma_S(w)(t) := J_0 \cdot w'(t) + S(t) \cdot w(t).$$

Then σ_S is a self-adjoint Fredholm operator of index zero. Thus σ_S defines an unbounded self-adjoint operator on $L^2(S^1, \mathbb{R}^{2n})$ with spectrum $\mathsf{spec}(\sigma_S)$ consisting of discrete eigenvalues of finite multiplicity that accumulate only at infinity. Moreover

$$\dim \ker \sigma_S = \dim \ker (\Psi(1) - \mathbb{1}). \tag{3.4}$$

Exercise 3.17. Prove Lemma 3.16. *Hint:* To show (3.4), note that if $\Psi(1)w_0 = w_0$ and $w(t) := \Psi(t)w_0$ then w is a loop that satisfes $\sigma_S(w) = 0$. See also the discussion on page 41.

We now define the class of trivialisations of the pullback bundles x^*TQ that we will use.

Definition 3.18. Let $x \in \Lambda Q$ and $J \in \mathcal{J}(Q,\omega)$. A *symplectic trivialisation* of the bundle $x^*TQ \to S^1$ is a map

$$\Phi : S^1 \times \mathbb{R}^{2n} \to x^*TQ,$$

written $(t,p) \mapsto \Phi_t(p)$, such that

$$\Phi_t^*\omega|_{x(t)} = \omega_0, \quad \Phi_t J_0 = J\Phi_t.$$

Moreover a trivialisation Φ is called *admissible* if

$$\Phi_t(p) = \overline{\Phi}_{e^{2\pi it}}(p)$$

for a symplectic trivialisation $\overline{\Phi} : \mathbb{D} \times \mathbb{R}^{2n} \to \overline{x}^*TQ$ of a capping $\overline{x} : \mathbb{D} \to Q$ of x. The next lemma is the reason why we require that the map $I_{c_1} : \pi_2(Q) \to \mathbb{Z}$ from Definition 2.22 vanishes.

Lemma 3.19. *For any $x \in \Lambda Q$ there always exists an admissible symplectic trivialisation Φ of x^*TQ. Moreover any two admissible symplectic trivialisations are homotopic.*

Proof. The existence of an admissible symplectic trivialisation can be proved using Gram-Schmidt, see [MS98, Lemma 2.65] for details. Two prove that any two such trivialisations are homotopic we proceed to two steps. First, suppose $\overline{x}$ is a capping and $\overline{\Phi}$ and $\overline{\Psi}$ are two trivialisations of $\overline{x}^*TQ$. Then $z \mapsto \overline{\Phi}_z^{-1} \circ \overline{\Psi}_z$ defines a map $\mathbb{D} \to \mathsf{U}(n,\mathbb{C})$. Every such map is smoothly homotopic to the constant map $z \mapsto \mathbb{1}_{2n}$. Now we prove that if $\overline{x}$ and $\overline{y}$ are two cappings with corresponding trivialisations $\overline{\Phi}$ and $\overline{\Psi}$ then the two trivialisations $\Phi := \overline{\Phi}|_{\partial\mathbb{D}}$ and $\Psi := \overline{\Psi}|_{\partial\mathbb{D}}$ are homotopic. Without loss of generality we may assume that there exists $\varepsilon > 0$ such that

$$\overline{x}(z) = \overline{x}(z/|z|) \quad \text{and} \quad \overline{\Phi}_z = \overline{\Phi}_{z/|z|}$$

for all $z \in \mathbb{D}$ with $1 - \varepsilon \leq |z| \leq 1$, and similarly for $\overline{y}$ and $\overline{\Psi}$. Then there is a well defined map

$$v : S^2 = \mathbb{C} \cup \{\infty\} \to Q$$

defined by

$$v(z) = \begin{cases} \overline{x}(z), & |z| \leq 1, \\ \overline{y}(1/|z|), & |z| \geq 1. \end{cases}$$

The bundle $v^*TQ \to S^2$ is trivial if and only if $c_1(v^*TQ) = 0$. Since we assume that $I_{c_1} \equiv 0$, this is indeed the case. Thus there exists a symplectic trivialisation $\Theta : S^2 \times \mathbb{R}^{2n} \to v^*TQ$. By restricting to the upper and lower hemispheres, we obtain two new trivialisations Θ^+ of $\overline{x}^*TQ$ and Θ^- of $\overline{y}^*TQ$. The argument above shows that $\overline{\Phi}$ is homotopic to Θ^+ and $\overline{\Psi}$ is homotopic to Θ^-. The claim follows. ∎

We will see later on why it is so important that any two admissible symplectic trivialisations are homotopic. For now let us simply note the following lemma.

Lemma 3.20. *A zero $x \in \mathcal{P}_1^\circ(H)$ is a regular zero of $\sigma_{J,H}$ in the sense of Definition 1.3 if and only if x is a non-degenerate element of $\mathcal{P}_1^\circ(H)$ in the sense of Definition 2.13.*

Proof. Pick an admissible symplectic trivialisation $\Phi : S^1 \times \mathbb{R}^{2n} \to x^*TQ$ of x. Let

$$\Psi(t) := \Phi_t^{-1} \circ D\varphi_H^t(x(0)) \circ \Phi_0$$

and let

$$S(t) := \Phi_t^{-1} \circ J(\nabla_t \Phi - \nabla_\Phi X_{H_t}(x)).$$

To prove that S is symmetric we define

$$V_j(t) := \Phi_t e_j,$$

where e_j is the jth standard basis vector in $\mathbb{R}^{2n}$. Abbreviate $X(t) := X_{H_t}(x(t))$ and $\langle \cdot, \cdot \rangle := g_J(\cdot, \cdot) = \omega(J\cdot, \cdot)$. If $S = [S_{ij}]$ then

$$S_{ij} = \left\langle V_i, J(\nabla_t V_j - \nabla_{V_j} X \right\rangle = \langle V_i, J[X, V_j] \rangle = \omega(V_i, [X, V_j]). \tag{3.5}$$

ecall that if α is a 1-form and β is a 2-form and U, V, W are vector fields then:

$$d\alpha(U, V) = U\alpha(V) - V\alpha(U) - \alpha([U, V]),$$

and

$$d\beta(U, V, W) = U\beta(V, W) + V\beta(W, U) + W\beta(U, V) - \beta([U, V], W) - \beta([V, W], U) - \beta([W, U], X).$$

Apply these formulae with $\alpha = dH$ and $\beta = \omega$ to discover that

$$V_i\omega(V_j, X) - V_j\omega(V_i, X) - \omega([V_i, V_j], X) = 0,$$

and

$$\begin{aligned} 0 &= V_i\omega(V_j, X) - V_j\omega(V_i, X) + X\omega(V_i, V_j) - \omega([V_i, V_j], X) - \omega([V_j, X], V_i) - \omega([X, V_i], V_j) \\ &= X\omega(V_i, V_j) - \omega([V_j, X], V_i) - \omega([X, V_i], V_j). \end{aligned}$$

Since the basis $\{V_i\}$ is a symplectic basis, $\omega(V_i, V_j)$ is constant and hence $X\omega(V_i, V_j) = 0$. Thus we have shown that

$$\omega(V_j, [X, V_i]) = \omega([V_j, X], V_i) = \omega(V_i, [X, V_j]),$$

and hence from (3.5) we see that S is symmetric as claimed.

Next we wish to show that $\dot{\Psi} = J_0 S \Psi$. To do this one differentiates the equation

$$\Phi_t \circ \Psi_t = D\varphi_H^t(x(0)) \circ \Phi_0$$

with respect to t to conclude that

$$\Phi_t \dot{\Psi}_t = J\nabla_t \Phi - J\nabla_\Phi X_{H_t}(x)\Psi_t,$$

and then since $\Phi_t \circ J_0 = J \circ \Phi_t$ we see that

$$\Phi_t J_0 \dot{\Psi}_t = J\Phi_t \dot{\Psi}_t = -\nabla_t \Phi + \nabla_\Phi X_{H_t}(x) = -\Phi_t S(t)\Psi_t.$$

Since Φ_t is an isomorphism the result finally follows. ∎

The main result of this section is the following theorem.

Theorem 3.21. *(Non-degeneracy is a generic property)*
There exists a subset $\mathcal{H}_{\mathsf{reg}} \subset C^\infty(S^1 \times Q)$ of second category with the property that if $H \in \mathcal{H}_{\mathsf{reg}}$ then every zero x of $\sigma_{J,H}$ is regular in the sense of Definition 1.3.

In order to prove Theorem 3.21 we will use the following result. Let $\mathcal{B}$ and $\mathcal{X}$ denote Banach manifolds and suppose $\mathcal{E} \to \mathcal{B} \times \mathcal{X}$ is a Banach bundle. Suppose $\sigma : \mathcal{B} \times \mathcal{X} \to \mathcal{E}$ is a section. We write

$$\sigma(b, x) = \sigma_b(x) = \sigma_x(b)$$

for $b \in \mathcal{B}$ and $x \in \mathcal{X}$.

Theorem 3.22 (The Transversality Theorem)**.** *Suppose σ is a section of class C^k. Assume that for all $x \in \mathcal{X}$ and all $b \in \sigma_x^{-1}(0)$, the vertical derivative*

$$D^v\sigma_x(b) : T_b\mathcal{B} \to \mathcal{E}_{(b,x)}$$

is a Fredholm operator of index l, with

$$k \geq \max\{1, 1 + l\}.$$

Suppose in addition for all $(b, x) \in \sigma^{-1}(0)$, the vertical derivative

$$D^v\sigma(b, x) : T_b\mathcal{B} \times T_x\mathcal{X} \to \mathcal{E}_{(x,b)}$$

is surjective. Then the set

$$\mathcal{X}_{\mathsf{reg}} := \left\{x \in \mathcal{X} \mid D^v\sigma_x(b) \text{ is surjective for each } b \in \sigma_x^{-1}(0)\right\}$$

is a residual (and hence dense) subset of $\mathcal{X}$.

In order to prove Theorem 3.22 we begin with the following linear result.

Proposition 3.23. *Let X, Y, X denote Banach spaces. Suppose $T : X \to Y$ is a Fredholm linear operator and $B : Z \to Y$ is a bounded linear operator. Then the range $\operatorname{ran} T \oplus B$ is a closed subspace of Y which admits a finite dimensional complement. If $T \oplus B$ is surjective then* ker $T \oplus B$ *admits a topological complement: that is, there exists a closed subspace $V \subset X \times Z$ such that*

$$X \times Z = (\ker\, T \oplus B) \oplus V, \quad (\ker\, T \oplus B) \cap V = \{(0, 0)\}.$$

Moreover the projection π onto the second factor:

$$\pi : \ker\, T \oplus B \to Z$$

is Fredholm, with $\ker\, \pi \cong \ker\, T$ *and* $\operatorname{coker} \pi \cong \operatorname{coker} T$*. In particular,*

$$\operatorname{ind} \pi = \operatorname{ind} T.$$

Proof. Firstly $\operatorname{ran} T$ is closed and $Y/\operatorname{ran} T$ is finite dimensional. Thus if $\operatorname{pr} : Y \to Y/\operatorname{ran} T$ denotes the projection operator then the subspace $\operatorname{pr}^{-1}(\operatorname{pr}(\operatorname{ran} B))$ is closed. This is precisely $\operatorname{ran} T \oplus B$. A complement W of $\operatorname{ran} T \oplus B$ is contained in the finite dimensional space $Y/\operatorname{ran} T$ and hence is closed.

Since ker T is finite dimensional it admits a topological complement X_1. Next, $\operatorname{ran} T$ is closed with finite dimensional complement $\operatorname{coker} T$, and thus surjectivity of $T \oplus B$ implies that $\operatorname{coker} T \subset \operatorname{ran} B$. Thus there exists a finite subset $\{z_1, \dots, z_k\}$ of Z such that $\{Bz_j\}$ is a basis of $\operatorname{coker} T$. Now define

$$S : Y = \operatorname{ran} T \oplus \operatorname{coker} T \to \ker T \oplus X_1 \oplus Z$$

by setting

$$S(y_1, y_2) := \left(0, x, \sum_{j=1}^{k} a_j z_j\right),$$

where $Tx = y_1$ and $y_2 = \sum_{j=1}^{k} a_j B z_j$. Then S is a right inverse of $T \oplus B$, and hence $\ker T \oplus B$ admits a topological complement.

For the last statement, set $K := \ker T \oplus B$, so that $\pi : K \to Z$. One easily checks that $\ker \pi = \ker T \oplus 0$. Finally,

$$\begin{aligned}
\operatorname{coker} \pi &= Z / \operatorname{ran} \pi \\
&= Z / B^{-1}(\operatorname{ran} T)) \\
&\cong \operatorname{ran} B / \operatorname{ran} T \cap \operatorname{ran} B \\
&\cong Y / \operatorname{ran} T = \operatorname{coker} T.
\end{aligned}$$

■

Next, we quote the following version of the classical Sard-Smale transversality theorem.

Theorem 3.24 (Sard-Smale transversality). *Suppose $\psi : \mathcal{M} \to \mathcal{N}$ is a C^k- map between separable Banach manifolds. Assume that ψ is Fredholm of index l, and suppose that $k \geq \max\{1, 1 + l\}$. Then the set of regular values of ψ is a residual (and hence dense) subset of $\mathcal{N}$.*

We can now prove Theorem 3.22.

Proof. (of Theorem 3.22)
We apply Proposition 3.23 with

$$T = D^v \sigma_x(b), \quad B = D^v \sigma_b(x).$$

Thus Proposition 3.23 implies that for any $(b, x) \in \sigma^{-1}(0)$ the vertical derivative $D^v \sigma(b, x)$ has a right inverse. Thus 0 is a regular value of x, and thus the Implicit Function Theorem 1.6 implies that the *universal moduli space* $\mathcal{M} := \sigma^{-1}(0)$ admits the structure of a C^k-Banach manifold.

Moreover if $\pi : \mathcal{M} \to \mathcal{X}$ denotes the projection then the last part of Proposition 3.23 tells us that $D\pi(b, x)$ is Fredholm for any $(b, x) \in \mathcal{M}$ and of index equal to that of $T = D^v \sigma_x(b)$. This means that we may apply the Sard-Smale Theorem 3.24 to conclude the regular values of π form a residual subset of $\mathcal{X}$. To complete the proof, we prove that $x \in \mathcal{X}$ is a regular value of π if and only if $D^v \sigma_b(x)$ is surjective for all $x \in \sigma_b^{-1}(0)$.

In other words, we must prove that

$$D\pi(b, x) \text{ is surjective } \Leftrightarrow \; D^v \sigma_x(b) \text{ is surjective for all } b \in \sigma_x^{-1}(0).$$

Suppose that $D^v \sigma_x(b)$ is surjective for all $b \in \sigma_x^{-1}(0)$. Fix $\hat{x} \in T_x \mathcal{X}$. Since $D^v \sigma_x(b)$ is surjective there exists $\hat{b} \in T_b \mathcal{B}$ such that

$$D^v \sigma_x(b)[\hat{b}] = D^v \sigma_b(x)[\hat{x}] \in \mathcal{E}_{(b,x)}.$$

Then

$$D^v\pi(b,x)[\hat{b},\hat{x}] = \hat{x}$$

and thus $D^v\pi(b,x)$ is surjective as required.

Conversely, suppose that $D^v\pi(b,x)$ is surjective. Fix $v \in \mathcal{E}_{(b,x)}$. Since $D^v\sigma(b,x)$ is surjective. there exists $(\hat{b},\hat{x})$ such that $D^v\sigma(\hat{b},\hat{x}) = v$. Thus

$$D^v\sigma_x(b)[\hat{b}] + D^v\sigma_b(x)[\hat{x}] = v.$$

Since $D^v\pi(b,x)$ is surjective there exists $\hat{b}_1$ such that

$$D^v\sigma_x(b)[\hat{b}_1] + D^v\sigma_b(x)[\hat{x}] = 0.$$

Thus if $\hat{b}_2 := \hat{b} - \hat{b}_1$ then

$$D^v\sigma_x(b)[\hat{b}_2] = v.$$

This completes the proof. ∎

Remark 3.25. In the setting of Theorem 3.22, since $\operatorname{ran} D^v\sigma(b,x)$ is necessarily closed (as $D^v\sigma_b(x)$ is bounded as $k \geq 1$ and $D^v\sigma_x(b)$ is Fredholm, cf. the proof of the first statement of Proposition 3.23 above), in order to show that $D^v\sigma(b,x)$ is surjective it is sufficient to show that annhilator of its image is zero. Thus it suffices to show that

$$(\operatorname{ran} D^v\sigma(b,x))^{\mathsf{ann}} = 0,$$

where

$$(\operatorname{ran} D^v\sigma(b,x))^{\mathsf{ann}} = \left\{\varphi \in \mathcal{E}^*_{(b,x)} \mid \varphi(v) = 0 \text{ for all } v \in \operatorname{ran} D^v\sigma(b,x)\right\}.$$

Exercise 3.26. Prove this using the Hahn-Banach theorem.

We will now prove Theorem 3.21.

Proof. (of Theorem 3.21)

The proof is an easy application of Theorem 3.22. We take $\mathcal{B} = \mathcal{L}Q$ and $\mathcal{X} = C^k(S^1 \times Q)$, and we take $\mathcal{E} \to \mathcal{L}Q \times C^k(S^1 \times Q)$ to be the Banach bundle whose fibre over (x,H) is simply $L^2(S^1, x^*TQ)$. For a given fixed almost complex structure J on Q that is ω-compatible, we define as before

$$\sigma(x,H) := \sigma_{J,H}(x) = J(x)(x'(t) - X_{H_t}(x(t))) = \nabla_J \mathbb{A}_H(x).$$

We have already shown that the map operator $D^v\sigma_H(x)$ is a Fredholm operator of index 0, and thus by Theorem 3.22 we need only show that $D^v\sigma(x,H)$ is surjective for all $(x,H) \in \sigma^{-1}(0)$. Moreover by Remark 3.25 it is sufficient to show that if $\hat{y} \in L^2(S^1, x^*TQ)$ has the property that

$$\int_{S^1} \left\langle D^v\sigma(x,H)[\hat{x},\hat{H}], \hat{y}\right\rangle_J dt = 0, \quad \text{for all } (\hat{x},\hat{H}) \in W^{1,2}(S^1, x^*TQ) \times C^k(S^1 \times Q)$$

then $\hat{y} \equiv 0$.

This is proved in two stages. Firstly, taking $\hat{H} = 0$ we see that

$$\int_{S^1} \langle D^v\sigma_H(x)[\hat{x}], \hat{y}\rangle_J \, dt = 0 \tag{3.6}$$

for all $\hat{x} \in W^{1,2}(S^1, x^*TQ)$, where we are using the fact that

$$D^v\sigma(x,H)[\hat{x},\hat{H}] = D^v\sigma_H(x)[\hat{x}] + D^v\sigma_x(H)[\hat{H}].$$

We claim that the fact that (3.6) holds for all $\hat{x}$ implies that $\hat{y} \in C^k(S^1, x^*TQ)$. To see this, it is convenient to go back to the local trivialisation picture used in the proof of Lemma 3.20. First note that x is certainly of class C^k, since $x' = X_{H_t}(x)$ with H of class C^k. If we choose a symplectic trivialisation $\Phi_t : \mathbb{R}^{2n} \to T_{x(t)}Q$ then if we define $v \in L^2(S^1, \mathbb{R}^{2n})$ by requiring that

$$\hat{y}(t) := \Phi_t(v(t))$$

then (3.6) becomes the assertion:

$$\int_{S^1} \langle J_0 w'(t) + S(t)\cdot w(t), v(t)\rangle \, dt = 0$$

for all $w \in W^{1,2}(S^1, \mathbb{R}^{2n})$, where S is a C^k-smooth path $[0,1] \to \mathsf{Sym}(\mathbb{R}^{2n})$. This equation says that v is a weak solution to the equation

$$\left(J_0 \frac{d}{dt} + S\right)[v] = 0,$$

where we are using the fact that S is symmetric. But clearly any weak solution is necessarily a strong solution, and hence we see that v is of class C^k. Thus so is $\hat{y}$. Now we know that $\hat{y}$ is of class C^k, it is easy to prove that $\hat{y} \equiv 0$. Indeed, suppose for contradiction that there exists $t_0 \in S^1$ such that $\hat{y}(t_0) \neq 0$. Choose $\hat{h} \in C^k(Q)$ such that

$$d\hat{h}(x(t_0))[\hat{y}(t_0)] > 0.$$

Since $\hat{y}$ is continuous, there exists $\epsilon > 0$ such that

$$d\hat{h}(x(t))[\hat{y}(t)] \geq 0, \quad \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon).$$

Choose a smooth cutoff function $\beta \in C^\infty(S^1, [0,1])$ such that $\beta(t_0) = 1$ and $\beta(t) = 0$ for $t \notin (t_0 - \epsilon, t_0 + \epsilon)$. Finally set

$$\hat{H}_t(x) := \beta(t)\hat{h}(x).$$

Then by construction we have

$$\int_{S^1} \left\langle D^v\sigma(x,H)[0,\hat{H}], \hat{y}\right\rangle_J dt = -\int_{S^1} d\hat{H}_t(x(t))[\hat{y}(t)]dt < 0,$$

which contradicts (3.6).

To finish the proof of the Theorem we need to go from the C^k-statement to the C^∞-infinity statement: we have shown that for each $k \in \mathbb{N}$ there exists a residual subset $\mathcal{H}^k_{\mathsf{reg}} \subset C^k(S^1 \times Q)$ such that if $H \in \mathcal{H}^k_{\mathsf{reg}}$ then all elements of $\mathcal{P}_1(H)$ are non-degenerate. Since $\mathcal{P}_1(H)$ is finite by the Exercise 3.28 below, it follows that $\mathcal{H}^k_{\mathsf{reg}}$ is not only a dense subset but also an *open* subset. Now by the Baire Category theorem we complete the proof with

$$\mathcal{H}_{\mathsf{reg}} := \bigcap_{k=1}^{\infty} \mathcal{H}^k_{\mathsf{reg}}.$$

■

Exercise 3.27. Suppose $H \in C^k(S^1 \times Q)$. Let $\varphi_H = \varphi_H^1$ denote the time-1 map. Let $x(t) \in \mathcal{P}_1(H)$ and set $q = x(0)$. Let $\Delta \subset Q \times Q$ denote the diagonal. Prove that x is a non-degenerate element of $\mathcal{P}_1(H)$ if and only if (q, q) is a transverse point of intersection of Δ and $\mathrm{graph}(\varphi_H)$.

Exercise 3.28. Prove that if $H \in C^k(S^1 \times Q)$ has the property that every element of $\mathcal{P}_1(H)$ is non-degenerate then $\mathcal{P}_1(H)$ is a finite set. *Hint:* Use the previous exercise.

CHAPTER 4

The Floer equation and elliptic regularity

From now on we will *always* assume our given Hamiltonian $H \in C^\infty(S^1 \times Q)$ belongs to $\mathcal{H}_{\mathsf{reg}}$. Fix once and for all an ω-compatible almost complex structure J on Q.

Suppose $u : \mathbb{R} \to \Lambda Q$ is a negative gradient flow line of $\mathbb{A}_H$. That is,

$$\frac{d}{ds}u(s) + \nabla_J \mathbb{A}_H(u(s)) = 0, \tag{4.1}$$

where $\nabla_J \mathbb{A}_H$ was defined in Definition 3.13. A map $u : \mathbb{R} \to \Lambda Q$ is the same thing as a map $u : \mathbb{R} \times S^1 \to Q$, and thought of in this way the equation (4.1), which is an ODE on the loop space, becomes a PDE on Q:

$$\partial_s u + J(u)\partial_t u - J(u)X_{H_t}(u) = 0. \tag{4.2}$$

Now we can explain Floer's brilliant observation. The point is, that the ODE (4.1) is a terrible ODE to try and do Morse theory with, as it is not well posed (cf. Remark 3.6). Nevertheless the PDE (4.2) *is* a nice equation to work with! Indeed, one can view (4.2) as a *perturbation* of the equation

$$\partial_s u + J(u)\partial_t u = 0,$$

which is the equation that u should satisfy in order to be a *J-holomorphic map*, as defined by Gromov [Gro85]. Gromov proved that J-holomorphic maps have lots of nice properties: for instance, typically they come in finite-dimensional families. Thus it is reasonable to hope that the perturbed equation (4.2) should also share most of these nice properties. This is indeed the case, and this is the key to why Floer homology works.

Definition 4.1. Fix $p > 2$ and consider the space $W^{1,p}(\mathbb{R} \times S^1, Q)$ of maps $u : \mathbb{R} \times S^1 \to Q$ that are locally of class $W^{1,p}$. This is the completion of the space $C^\infty(\mathbb{R} \times S^1, Q)$ in the Sobolev $W^{1,p}$-norm. Note we really need $p > 2$: since $\mathbb{R} \times S^1$ is 2-dimensional, elements of Sobolev class $W^{1,p}$ are continuous only when $p > 2$. Thus the space $W^{1,p}(\mathbb{R} \times S^1, Q)$ is only well defined when $p > 2$!

Remark 4.2. For $p \leq 2$ one can still define this space by choosing an embedding $Q \hookrightarrow \mathbb{R}^N$ for some N and then setting

$$W^{1,p}(\mathbb{R} \times S^1, Q) := \left\{ u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^N) \mid u(s,t) \in Q \text{ for a.e. } (s,t) \right\}.$$

Nevertheless for $p \leq 2$ this space depends on the specific choice of embedding $Q \hookrightarrow \mathbb{R}^N$.

Let $\mathcal{E}^p$ denote the Banach bundle over $W^{1,p}(\mathbb{R} \times S^1, Q)$ whose fibre over u is the space

$$\mathcal{E}^p_u = L^p(\mathbb{R} \times S^1, u^*TQ). \tag{4.3}$$

Definition 4.3. The *Floer operator*

$$\overline{\partial}_{J,H} : W^{1,p}(\mathbb{R} \times S^1, Q) \to \mathcal{E}$$

is defined by

$$\overline{\partial}_{J,H}(u) := \partial_s u + J(u)\partial_t u - J(u)X_{H_t}(u).$$

The aim in the next few lectures is to prove that the linearization $D^v\overline{\partial}_{J,H}(u)$ at a zero u is a Fredholm operator, and that (after a small perturbation of H away from the elements of $\mathcal{P}_1(H)$), every zero of $\overline{\partial}_{J,H}$ is regular. This will allow us to define the moduli spaces used in Floer homology. As in the definition of the spaces $\mathcal{B}(x,y)$ on page 4, in order to get a nice Fredholm operator we need to restrict $\overline{\partial}_{J,H}$ to an appropriate path space.

Fix a 'background' Riemannian metric g on Q. This metric will be used to define the exponential map exp in the next definition. Since Q is compact, any two Riemannian metrics on Q define the same topology, and hence the choice of g makes no difference to the spaces $\mathcal{B}^{1,p}(x^-,x^+)$ defined below.

Definition 4.4. Fix $x^-,x^+ \in \mathcal{P}_1(H)$. We define the space

$$\mathcal{B}^{1,p}(x^-,x^+) \subset W^{1,p}(\mathbb{R}\times S^1, Q)$$

to be the subset of maps $u \in W^{1,p}(\mathbb{R}\times S^1,Q)$ with the following additional property: there exists $s_0 > 0$ and sections

$$\xi^- \in W^{1,p}((-\infty,-s_0]\times S^1,(x^-)^*TQ), \quad \xi^+ \in W^{1,p}([s_0,+\infty)\times S^1,(x^+)^*TQ)$$

such that for $|s| \geq s_0$ one has

$$u(s,t) = \exp_{x^{\pm}(t)}(\xi^{\pm}(s,t)). \tag{4.4}$$

Exercise 4.5. Convince yourself that the space $\mathcal{B}^{1,p}(x^-,x^+)$ carries the structure of a Banach manifold, and that

$$T_u\mathcal{B}^{1,p}(x^-,x^+) = W^{1,p}(\mathbb{R}\times S^1,u^*TQ).$$

Moreover the bundle $\mathcal{E}^p$ defined in (4.3) restricts to give a well defined Banach bundle $\mathcal{E}^p \to \mathcal{B}^{1,p}(x^-,x^+)$. Finally, prove that

$$\overline{\partial}_{J,H} : \mathcal{B}^{1,p}(x^-,x^+) \to \mathcal{E}^p$$

is a smooth section. *Hint:* Ignore this exercise. It is very boring and tedious.

In order to get any further we will need to recall two elliptic results, which we will quote without proof. Let J_0 denote the standard almost complex structure on $\mathbb{R}^{2n}$ defined in (**??**), and let

$$\overline{\partial} := \partial_s + J_0\partial_t$$

denote standard *Cauchy-Riemann operator* on $\mathbb{R}^{2n}$.

Theorem 4.6 (The Calderon-Zygmund inequality)**.** *Let $1 < p < +\infty$. There exists a constant $c(p,n) > 0$ such that*

$$\|\nabla u\|_{L^p(\mathbb{C})} \leq c(p,n)\left\|\overline{\partial}u\right\|_{L^p(\mathbb{C})}$$

for every compactly supported smooth map $u : \mathbb{C} \to \mathbb{R}^{2n}$.

This theorem is highly non-trivial. See for instance [MS12, Appendix B] for a detailed proof. A density argument then gives the following result, which we leave as an exercise:

Exercise 4.7. Let $1 < p < +\infty$. There exists a constant $c(p,n) > 0$ such that for every map $u \in W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ it holds that

$$\|\nabla u\|_{L^p(\mathbb{R}\times S^1)} \leq c(p,n)\left\|\overline{\partial}u\right\|_{L^p(\mathbb{R}\times S^1)}. \tag{4.5}$$

Definition 4.8. Suppose $\Omega \subset \mathbb{C}$ is an open domain and $u, f \in L^1_{\mathrm{loc}}(\Omega)$. Recall we say that u is a *weak solution* to the equation

$$\overline{\partial} u = f$$

if for every $\varphi \in C^\infty_c(\mathbb{C})$ one has

$$\int_\Omega \langle u, \partial\varphi \rangle \, dsdt = -\int_\Omega \langle f, \varphi \rangle \, dsdt,$$

where here

$$\partial := \partial_s - J_0 \partial_t.$$

Theorem 4.9 (Local regularity for weak solutions)**.** *Suppose $\Omega \subset \mathbb{C}$ is an open domain and $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$. Then if $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to the equation $\overline{\partial} u = f$ then $u \in W^{k+1,p}_{\mathrm{loc}}(\Omega)$. Moreover, if $U \subset \Omega$ is an open domain such that $\overline{U} \subset \Omega$ then there exists a constant $c = c(p, k, U, \Omega) > 0$ such that*

$$\|u\|_{W^{k+1,p}(\overline{U})} \le c \left(\|f\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)} \right). \tag{4.6}$$

In particular, if f is smooth then so is u.

This is actually a (fairly) easy consequence of Theorem 4.6.

Exercise 4.10. Use Theorem 4.6 to prove Theorem 4.9. *Hint:* See [MS12, Appendix B] if you get stuck.

Let us now fix two critical points $x^-, x^+ \in \mathcal{P}_1(H)$, and assume $u \in \mathcal{B}^{1,p}(x^-, x^+)$ satisfies $\overline{\partial}_{J,H}(u) = 0$. We would like to prove the following *elliptic regularity* result:

Theorem 4.11 (Elliptic regularity)**.** *If $\overline{\partial}_{J,H}(u) = 0$ then $u \in C^\infty(\mathbb{R} \times S^1, Q)$.*

This result will take a long time to prove. We will begin by proving a linear version:

Theorem 4.12. *Let $1 < p < \infty$ and suppose*

$$S \in C^\infty(\mathbb{R} \times S^1, \mathsf{L}(\mathbb{R}^{2n})) \cap L^\infty(\mathbb{R} \times S^1, \mathsf{L}(\mathbb{R}^{2n}))$$

is a smooth map such that the limits $S^\pm(t) := \lim_{s \to \pm\infty} S(s,t)$ exist and such that

$$\lim_{s \to \pm\infty} \frac{\partial S}{\partial s}(s,t) = 0,$$

uniformly in t. Let

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

denote the perturbed Cauchy-Riemann operator

$$D_S u := \partial_s u + J_0 \partial_t u + Su.$$

Then:

1. *Suppose $u \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is a weak solution of $D_S u = 0$. Then $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \cap C^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.*

2. *If $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ then there exists a constant $c = c(p) > 0$ such that*

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c\left(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p(\mathbb{R}\times S^1)}\right). \tag{4.7}$$

Remark 4.13. Strictly speaking we do not need to prove (4.7) in order to prove Theorem 4.11. However (4.7) will be useful when it comes to showing that $D^v\overline{\partial}_{J,H}(u)$ is a Fredholm operator.

Proof. The proof of (1) is an easy consequence of Theorem 4.9. Indeed, if $u \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is a weak solution to the equation $D_S u = 0$ then u is a weak solution of

$$\overline{\partial} u = -Su,$$

where $\overline{\partial} = \partial_s + J_0\partial_t$. Note that $Su \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Theorem 4.9 then implies that $u \in W^{1,p}_{\text{loc}}$. But then $Su \in W^{1,p}_{\text{loc}}$. Thus another application of Theorem 4.9 implies that $u \in W^{2,p}_{\text{loc}}$. By induction, we see that $u \in W^{k,p}_{\text{loc}}$ for all $k \in \mathbb{N}$, and hence $u \in C^\infty(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ by the Sobolev Embedding Theorem.

To prove (2), we first note that the Calderon-Zygmund inequality (Theorem 4.6) implies that if $u \in C^\infty_c(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ then there exists a constant $c_1 > 0$ such that

$$\|\nabla u\|_{L^p(\mathbb{R}\times S^1)} \leq \|\overline{\partial} u\|_{L^p(\mathbb{R}\times S^1)},$$

and hence

$$\begin{aligned}\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} &\leq c_1\left(\|\overline{\partial} u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p(\mathbb{R}\times S^1)}\right)\\ &\leq c_1\left(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \left(1 + \|S\|_{L^\infty(\mathbb{R}\times S^1)}\right)\|u\|_{L^p(\mathbb{R}\times S^1)}\right)\\ &\leq c\left(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p(\mathbb{R}\times S^1)}\right),\end{aligned}$$

where $c = c_1\left(1 + \|S\|_{L^\infty(\mathbb{R}\times S^1)}\right)$. ■

Here is an extension of Theorem 4.9. Note that if $\overline{\partial} u = f$ then

$$\Delta u = \partial\overline{\partial} u = \partial f.$$

Lemma 4.14. *Suppose $\Omega \subset \mathbb{R} \times S^1$ is an open domain and $u, f, g, h, \in L^p_{\text{loc}}(\Omega)$, and suppose that u is a weak solution of the equation:*

$$\Delta u = f + \partial_s g + \partial_t h$$

(where $\partial_s g$ and $\partial_t h$ are interpreted as distributions). Then in fact $u \in W^{1,p}_{\text{loc}}(\Omega)$. Moreover if $U \subset \Omega$ is an open domain with $\overline{U}$ compact, $\overline{U} \subset \Omega$, then there exists a constant $c = c(p, U, \Omega)$ such that

$$\|u\|_{W^{1,p}(U)} \leq c\left(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right).$$

Exercise 4.15. Prove Lemma 4.14.

We now move onto the proof of the non-linear version, Theorem 4.11. By choosing a finite atlas of charts on Q, we may assume $Q = \mathbb{R}^{2n}$.

Remark 4.16. *Warning:* Do not confuse 'linear' and 'working on $\mathbb{R}^{2n}$'. As we shall see, the non-linear version is much harder, despite the fact that superficially they look the same.

We start from the assumption that we are given a $W^{1,p}_{\mathrm{loc}}$ map $u : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ satisfying

$$\partial_s u + J(u)\partial_t u - J(u)X_{H_t}(u) = 0.$$

Set

$$J(s,t) := J(u(s,t)), \quad f(s,t) := J(u(s,t))X_H(u(s,t)).$$

Thus $J \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}\times S^1, \mathsf{L}(\mathbb{R}^{2n}))$ and $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$. The first step is the following lemma:

Lemma 4.17. *Suppose $U \subset \Omega \subset \mathbb{R}\times S^1$ are open domains with compact closure, with $\overline{U} \subset \Omega$. Suppose $p > 2$ and $q, r > 0$ with $r > 1$ are such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{4.8}$$

Suppose $J \in W^{1,p}(\Omega, \mathsf{L}(\mathbb{R}^{2n}))$ satisfies $J^2 = -\mathbb{1}$ and

$$\|J\|_{W^{1,p}(\Omega)} \le c_0.$$

Suppose $f \in L^r_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$. If $u \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ is a weak solution of the equation

$$\partial_s u + J\partial_t u = f \tag{4.9}$$

then in fact $u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$. Moreover there exists a constant $c = c(c_0, U, \Omega) > 0$ such that

$$\|u\|_{W^{1,r}(U)} \le c\left(\|f\|_{L^r(\Omega)} + \|u\|_{L^q(\Omega)}\right).$$

Proof. We will use Lemma 4.14. We apply $\partial_s - J\partial_t$ to (4.9) to obtain

$$(\partial_s - J\partial_t)(\partial_s + J\partial_t)u = (\partial_s - J\partial_t)f,$$

which gives

$$\begin{aligned}
\Delta u &= J\cdot\partial_t J\cdot\partial_t u - \partial_s J\cdot\partial_t u + \partial_s f - J\cdot\partial_t f\\
&= J\cdot\partial_t J\cdot(J\cdot\partial_s u - Jf) - \partial_s J\cdot\partial_t u + \partial_s f - J\cdot\partial_t f\\
&\overset{(*)}{=} -J^2\partial_t J\cdot(\partial_s u - f) - \partial_s J\cdot\partial_t u + \partial_s f - J\cdot\partial_t f\\
&= -\partial_s J\cdot\partial_t u - \partial_t J(f - \partial_s u) + \partial_s f - J\cdot\partial_t f\\
&\overset{(**)}{=} \partial_s(\partial_t J\cdot u + f) + \partial_t(-\partial_s J\cdot u - Jf),
\end{aligned}$$

where $(*)$ used the fact that $\partial_t J\cdot J = -J\cdot\partial_t J$ (differentiate $J^2 = \mathbb{1}$ with respect to t), and $(**)$ used the fact that $\partial^2_{st}J = \partial^2_{ts}J$ (even as weak solutions!). Next, Hölder's inequality implies that both $g := \partial_t J\cdot u + f$ and $h := -\partial_s J\cdot u - Jf$ belong to $L^r_{\mathrm{loc}}(\Omega)$:

$$\|g\|_{L^r(\Omega)} \le \|\partial_t J\|_{L^p(\Omega)}\|u\|_{L^q(\Omega)} + \|f\|_{L^r(\Omega)}, \quad \|h\|_{L^r(\Omega)} \le \|\partial_s J\|_{L^p(\Omega)}\|u\|_{L^q(\Omega)} + \|Jf\|_{L^r(\Omega)},$$

where we have used the fact that J is continuous as $p > 2$ (this is where we use (4.8)). Thus by Lemma 4.14 we conclude that

$$\begin{aligned}
\|u\|_{W^{1,r}(U)} &\le c_1\left(\|\partial_t J\cdot u\|_{L^r(\Omega)} + \|f\|_{L^r(\Omega)} + \|\partial_s J\cdot u\|_{L^r(\Omega)} + \|Jf\|_{L^r(\Omega)}\right)\\
&\le c_1\left(\|J\|_{W^{1,p}(\Omega)}\|u\|_{L^q(\Omega)} + \|f\|_{L^r(\Omega)} + \|J\|_{W^{1,p}(\Omega)}\|u\|_{L^q(\Omega)} + \|J\|_{L^\infty(\Omega)}\|f\|_{L^r(\Omega)}\right)\\
&\le c_1\left(2c_0\|u\|_{L^q(\Omega)} + (1 + Lc_0)\|f\|_{L^r(\Omega)}\right),
\end{aligned}$$

where L is a constant such that $\|\cdot\|_{L^\infty} \le L\|\cdot\|_{W^{1,p}}$ (such L exist since $p > 2$ and hence $W^{1,p}$ embeds compactly in L^∞). This proves the lemma. ■

Exercise 4.18. Check that Lemma 4.17 still holds if $q = +\infty$, with $r = p < +\infty$.

The following result completes the proof of Theorem 4.11. Next we prove the following result:

Theorem 4.19. *Suppose $k \geq 1$ and $p > 2$. Suppose $J \in W^{k,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathsf{L}(\mathbb{R}^{2n}))$ satisfies $J^2 = -\mathbb{1}$ and $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. If $u \in L^p_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is a weak solution of the equation*

$$\partial_s u + J \partial_t u = f$$

then $u \in W^{k+1,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Moreover if $U \subset \Omega \subset \mathbb{R} \times S^1$ are open domains with compact closure, with $\overline{U} \subset \Omega$, and

$$\|J\|_{W^{k,p}(\Omega)} \leq c_0,$$

then there exists a constant $c = c(p, n, k, c_0, U, \Omega) > 0$ such that for every $0 \leq j \leq k-1$ one has

$$\|u\|_{W^{j+1,p}(U)} \leq c \left(\|f\|_{W^{j,p}(U)} + \|u\|_{W^{j,p}(U)} \right).$$

Proof. Let us begin with the case $k = 1$. Our goal therefore is to show that $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. As before we apply $\partial_s - J\partial_t$ to discover that

$$\Delta u + \partial_s J \cdot u - J \cdot \partial_t J \cdot u - \partial_s f + J \cdot \partial_t f = 0.$$

Our first goal is to show that $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. If we could show that $u \in L^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ then we coudl apply Lemma 4.14 with $r = p$ and $q = +\infty$ (c.f. Exercise 4.18). Unfortunately we do not yet know this is the case. So we first show that $u \in W^{1,r}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ for some $r > 2$. Then by the Sobolev Embedding Theorem $u \in L^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

Unfortunately one cannot simply apply Lemma 4.14, since $\partial_s J \cdot u$ only belongs to $L^{p/2}_{\mathrm{loc}}$, and so unless $p > 4$ we cannot use Lemma 4.14 to deduce that $u \in W^{1,r}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ for some $r > 2$. Instead, the idea is to successively apply Lemma 4.17 for a clever set of choice of triples (p_k, q_k, r_k) satisfying $p_k > 2$, $r_k > 1$ and $1/p_k + 1/q_k + 1/r_k = 1$.

Claim: There exists a finite sequence of tuples (p_k, q_k, r_k), for $0 \leq k \leq l+1$, such that:

- $p_0 = q_0 = p$, ,
- $1 < r_0 < r_1 < \cdots < r_l < 2 < r_{l+1} < p$,
- for $1 \leq k \leq l+1$, $q_k = \frac{2r_{k-1}}{2 - r_{k-1}}$.

To prove the existence of such a sequence one chooses $1 < r_0 < p/2$ and defines inductively

$$r_{k+1} = \frac{2pr_k}{2p + 2r_k - pr_k}.$$

If $r_k < 2$ then $r_{k+1} > r_k$. If $r_k < 2$ for all k then the sequence would converge to a limit $r_\infty \leq 2$, which contradicts the recurrence relation. Thus there exists a finite l such that $r_l < 2 < r_{l+1}$.

Now choose a sequence of open domains

$$U \subset U_{l+1} \subset U_k \subset \cdots \subset U_0 \subset \Omega,$$

such that at each stage one has $\overline{U}_{k+1}$ compact and $\overline{U}_{k+1} \subset U_k$. The Sobolev Embedding Theorem tells us that

$$L^p(U_{k-1}) \subset L^{r_k}(U_{k-1}),$$
$$W^{1,r_{k-1}}(U_{k-1}) \subset L^{q_k}(U_{k-1})$$

(this was the reason for requiring $q_k = \frac{2r_{k-1}}{2-r_{k-1}}$!). Moreover there exists a constant b_k such that

$$\|\cdot\|_{L^{q_k}(U_{k-1})} \leq b_k \|\cdot\|_{W^{1,r_{k-1}}(U_{k-1})}.$$

So now let us apply Lemma 4.17 with (p_0, q_0, r_0, U_0). This tells us that $u \in W^{1,r_0}(U_0)$, and moreover that there exists a constant k_0 such that

$$\|u\|_{W^{1,r_0}(U_0)} \leq a_0 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right).$$

Now apply Lemma 4.17 with (p_1, q_1, r_1, U_1). We see that $u \in W^{1,r_1}(U_1)$ and obtain

$$\|u\|_{W^{1,r_1}(U_1)} \leq a_1 \left(\|f\|_{L^{r_1}(U_0)} + \|u\|_{L^{q_1}(U_0)} \right) \leq a_1 b_1 \left(\|f\|_{L^p(\Omega)} + \|u\|_{W^{1,r_0}(U_0)} \right).$$

Now we repeatedly apply Lemma 4.17 another l times to see that $u \in W^{1,r_{l+1}}(U_{k+1})$ and moreover that

$$\|u\|_{W^{1,r_{l+1}}(U_{l+1})} \leq a \left(\|f\|_{L^p(U)} + \|u\|_{W^{1,r_{l+1}}(U_{l+1})} \right).$$

But now we are finally in good shape! Since $r_{l+1} >2$, one has $W^{1,r_{l+1}}(U_{l+1}) \subset L^\infty(U_{l+1})$, and hence we now know that $u \in L^\infty(U_{l+1})$. Now we apply Lemma 4.17 one more time with $(p, +\infty, p, U_{l+1})$ to deduce that $u \in W^{1,p}(U)$ and that there exists a constant $c > 0$ such that

$$\|u\|_{W^{1,p}(U)} \leq c \left(\|f\|_{L^p(U)} + \|u\|_{L^p(U)} \right).$$

This game is called *elliptic bootstrapping*. We are not done yet though! The next step is to show that actually $u \in W^{2,p}(U)$. For this one plays exactly the same game again with $v = \partial_s u$ and $w = \partial_t u$. One has

$$\partial_s v + J \partial_t v - g = 0,$$

where

$$g = \partial_s f - \partial_s J \cdot \partial_t u.$$

Unfortunately we are even worse off than before, since g is only in $L^{p/2}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Nevertheless by applying Lemma 4.17 with $p = q$ and $r = p/2$ we see that $v \in W^{1,p/2}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, and hence in $L^{q_1}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, where $q_1 = 2r/2 - r$. Similarly $w \in L^{q_1}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Thus if r_1 is defined by $1/p + 1/q_1 = 1/r_1$ then we see that $g \in L^{r_1}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Now we can repeat the arguments given above to show that both v and w belong to $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

We have now finally proved the case $k = 1$ of Theorem 4.19. One now argues inductively, as the following two exercises show. ■

Exercise 4.20. Prove that if $p > 2$ and $k \geq 1$ then if $f, g \in W^{k,p}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ then the product fg also belongs to $W^{k,p}_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, and moreover if $\Omega \subset \mathbb{R} \times S^1$ is an open domain with compact closure then there exists a constant $c > 0$ such that

$$\|fg\|_{W^{k,p}(\Omega)} \leq c \|f\|_{W^{k,p}(\Omega)} \|g\|_{W^{k,p}(\Omega)}.$$

Exercise 4.21. Complete the proof of Theorem 4.19. *Hint:* You will need to use Exercise 4.20. See [AD10, p430-432] if you get stuck.

We have now completed the proof of Theorem 4.11.

CHAPTER 5

Fredholm theory

The aim of this section is to prove that if $u \in \mathcal{B}^{1,p}(x^-, x^+)$ satisfies $\overline{\partial}_{J,H}(u) = 0$ then the vertical derivative $D^v\overline{\partial}_{J,H}(u)$ is a Fredholm operator. The first step is to choose a symplectic trivialisation of the pullback bundle $u^*TQ \to \mathbb{R} \times S^1$. For this we use the following lemma, which is an extension of the first part of Lemma 3.19.

Lemma 5.1. *A symplectic vector bundle $E \to \Sigma$ over a compact Riemann surface Σ with non-empty boundary $\partial\Sigma$ admits a symplectic trivialisation. Moreover the restriction of the symplectic trivialisation to any of the boundary circles is admissible in the sense of Definition 3.18.*

Exercise 5.2. Prove Lemma 5.1. *Hint:* Set $k(\Sigma) := 2g(\Sigma) + \#\pi_0(\partial\Sigma)$ and preceed by induction over $k(\Sigma)$. See [MS12, Proposition 2.66] if you get stuck.

Strictly speaking we cannot directly apply Lemma 5.1 to the symplectic bundle $u^*TQ \to \mathbb{R}\times S^1$, since $\mathbb{R}\times S^1$ is *not* a compact Riemann surface with non-empty boundary. However since we already know that u is a smooth map, we can employ the following trick.

Denote by $\overline{\mathbb{R}} = [-\infty, +\infty]$ the compactification of $\mathbb{R}$, equipped with the structure of a compact bounded manifold by the requirement that $s \mapsto s(1 + s^2)^{-1/2}$ defines a diffeomorphism from $\overline{\mathbb{R}}$ to $[-1, 1]$.

Exercise 5.3. Suppose $u \in \mathcal{B}^{1,p}(x^-, x^+) \cap C^\infty(\mathbb{R} \times S^1, Q)$. Show that u extends to define a smooth map $\overline{u} : \overline{\mathbb{R}} \times S^1 \to Q$ satisfying $\overline{u}(\pm\infty, t) = x^\pm(t)$.

Thus applying Lemma 5.1 to the compactified map $\overline{u}$, we obtain a symplectic trivialisation

$$\overline{\Phi} : \overline{\mathbb{R}} \times S^1 \to \overline{u}^*TQ$$

with the property that $\Phi := \overline{\Phi}|_{\mathbb{R}\times S^1}$ defines a symplectic trivialisation of u^*TQ, and $\Phi^\pm := \Phi|_{\{\pm\infty\}\times S^1}$ defines an admissible symplectic trivialisation of the asymptotic bundles $(x^\pm)^*TQ \to S^1$. As in the proof of Lemma 3.20 we now consider the smooth family $S : \mathbb{R} \times S^1 \to \mathsf{L}(\mathbb{R}^{2n})$ of matrices defined by and let

$$S(s,t) := \Phi_{s,t}^{-1} \circ (\nabla_s\Phi + J(\nabla_t\Phi - \nabla_\Phi X_{H_t}(u))\,. \tag{5.1}$$

The limit matrices

$$S^\pm(t) := \lim_{s\to\pm\infty} S(s,t) = (\Phi_t^\pm)^{-1} \circ J(\nabla_t\Phi^\pm - \nabla_{\Phi^\pm} X_{H_t}(x^\pm)$$

are symmetric (cf. the proof of Lemma 3.20). Since $\Phi_{s,t} : \mathbb{R}^{2n} \to T_{u(s,t)}Q$ is an isomorphism for each $(s,t) \in \mathbb{R} \times S^1$, we obtain that:

Lemma 5.4. *The operator $D\overline{\partial}_{J,H}(u) : W^{1,p}(\mathbb{R} \times S^1, u^*TQ) \to L^p(\mathbb{R} \times S^1, u^*TQ)$ is Fredholm if and only if the operator $D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ defined as usual by*

$$D_S u = \partial_s u + J_0\partial_t u + Su$$

is Fredholm. Moreover

$$\operatorname{ind} D\overline{\partial}_{J,H}(u) = \operatorname{ind} D_S.$$

Exercise 5.5. Prove Lemma 5.4.

The main result of this section is the following result:

Theorem 5.6. *Fix $1 < p < +\infty$ and suppose $S : \mathbb{R} \times S^1 \to \mathsf{L}(\mathbb{R}^{2n})$ is a smooth map such that the limits $S^{\pm}(t) := \lim_{s \to \pm\infty} S(s,t)$ exist and the convergence is uniform in t. Let*

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

denote the perturbed Cauchy-Riemann operator

$$D_S u := \partial_s u + J_0 \partial_t u + S u.$$

Suppose that the limit operators $S^{\pm}$ are symmetric matrices for each $t \in S^1$, and moreover suppose that they are non-degenerate in the sense that the fundamental solutions $\Psi^{\pm} : [0,1] \to \mathsf{Sp}(\mathbb{R}^{2n}, \omega_0)$ of $S^{\pm}$ (i.e. the solutions of the equation

$$(\Psi^{\pm})'(t) = J_0 \cdot S^{\pm}(t) \cdot \Psi^{\pm}(t), \quad \Psi^{\pm}(0) = \mathbb{1},$$

cf. Lemma 3.16) satisfy $\det(\Psi^{\pm}(1) - \mathbb{1}) \neq 0$. Then the operator D_S is Fredholm.

Later on in Theorem 6.21 we will compute the index of D_S, but for now we will content ourselves with showing only that D_S is Fredholm. The proof consists of two steps. In the first step we show that if $S(s,t)$ does not depend on s, then the operator D_S is not only Fredholm but is in fact an isomorphism.

Theorem 5.7. *Fix $1 < p < +\infty$ and suppose $S : S^1 \to \mathsf{Sym}(\mathbb{R}^{2n})$. Let*

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

denote the perturbed Cauchy-Riemann operator

$$D_S u := \partial_s u + J_0 \partial_t u + S u.$$

Assume that S is non-degenerate. Then the operator D_S is invertible.

We will first prove Theorem 5.7 in the case $p = 2$. The proof will require the following classical result, which is a special case of the Hille-Yosida theorem (see for instance [Rud91, Theorem 13.37]).

Theorem 5.8. *Let H be a Hilbert space and A an unbounded self-adjoint linear operator. Assume that $\operatorname{spec} A \subset (-\infty, \kappa]$ for some $\kappa \in \mathbb{R}$. Then there exists a family of bounded operators $E_A(s) : s \in [0, +\infty)$ such that:*

1. *$\lim_{s \downarrow 0} E_A(s)x = x$ for all $x \in H$,*
2. *$E_A(s+t) = E_A(s) \circ E_A(t)$ for all $s, t \geq 0$,*
3. *$E_A(s)(H) \subset \operatorname{dom} A$ for every $s > 0$,*
4. *the map $s \mapsto E_A(s)$ belongs to $C^\infty([0, +\infty), \mathsf{L}(H))$,*
5. *$\frac{d}{ds} E_A(s) = A \circ E_A(s)$,*
6. *$\|E_A(s)\|_{\mathsf{L}(H)} \leq e^{\kappa s}$ for all $s \geq 0$.*

One often writes $E_A(s) = \exp(sA)$.

In order to prove Theorem 5.7 in the case $p = 2$, let us set

$$\mathcal{H} := L^2((0,1), \mathbb{R}^{2n}), \quad \mathcal{W} := W^{1,2}(S^1, \mathbb{R}^{2n}).$$

Then

$$L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n}) = L^2(\mathbb{R}, \mathcal{H}), \quad W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) = W^{1,2}(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{W}),$$

where the norm of the intersection of two spaces is by definition the sum of the two norms. Now consider the unbounded linear operator A on $\mathcal{H}$ with dense domain $\mathcal{W}$ given by

$$A = J_0 \partial_t + S.$$

(thus A was denoted by σ_S in Lemma 3.16). In fact, as claimed in Lemma 3.16, A is symmetric. To see this one computes using integration by parts and the fact that $J_0^* = -J_0$ that:

$$\langle Au, v \rangle_{\mathcal{H}} = \int_0^1 (J_0 u' + Su) \cdot v = \int_0^1 u \cdot (J_0 v' + Sv) dt = \langle u, Av \rangle_{\mathcal{H}}.$$

Next, we claim that the assertion that S is non-degenerate implies that A has a bounded inverse. In other words, given $v \in \mathcal{H}$ we must find a unique $u \in \mathcal{W}$ such that $Au = v$. Let Ψ denote the fundamental solution S, so that $\Psi' = J_0 S \Psi$. Then if $Au = v$ then $u' = J_0 Su - J_0 v$, and hence given $\xi \in \mathbb{R}^{2n}$, the solution u_ξ of $Au_\xi = v$ satisfying $u_\xi(0) = \xi$ is given by

$$u_\xi(t) = \Psi(t) \left(\xi - \int_0^t \Psi(s)^{-1} J_0 v(s) ds \right), \quad \text{for all } \xi \in \mathbb{R}^{2n}.$$

One has $u_\xi \in \mathcal{W}$ if and only if $u_\xi(1) = 1$, which is the case if and only if

$$(\Psi(1) - \mathbb{1})\xi = \Psi(1) \int_0^1 \Psi(s)^{-1} J_0 v(s) ds.$$

Thus the inverse $A^{-1} : \mathcal{H} \to \mathcal{H}$ is the compact operator

$$A^{-1}v(t) := \Psi(t) \left((\Psi(1) - \mathbb{1})^{-1} \Psi(1) \int_0^1 \Psi(s)^{-1} J_0 v(s) ds - \int_0^t \Psi(s)^{-1} J_0 v(s) ds \right).$$

Since A is symmetric and invertible, A is self adjoint. So its spectrum is real, and by the spectral theorem we can decompose $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ in the positive and negative eigenspaces of A. Thus if

$$A^\pm := A|_{\mathcal{H}^\pm \cap \mathcal{W}}$$

then $A^\pm$are self-adjoint unbounded linear operators with domain $\operatorname{dom} A^\pm = \mathcal{H}^\pm \cap \mathcal{W}$. Moreover there exists $\kappa > 0$ such that $\operatorname{spec} A^+ \subset [\kappa, +\infty)$ and $\operatorname{spec} A^- \subset (-\infty, -\kappa]$. Thus by Theorem 5.8 the operators $-A^+$ and A^- generate families $E_{-A^+}(s), E_{A^-}(s) \in \mathsf{L}(\mathcal{H}^\pm)$. Let $P^\pm : \mathcal{H} \to \mathcal{H}^\pm$denote the orthogonal projections onto $\mathcal{H}^\pm$ and consider the discontinuous path of bounded linear operators given by

$$G(s) := \begin{cases} E_{-A^+}(s) P^+, & s \geq 0, \\ -E_{A^-}(-s)^- P^-, & s < 0. \end{cases}$$

Then

$$\|G\|_{\mathsf{L}(\mathcal{H})} \leq e^{-\kappa|s|}, \tag{5.2}$$

and we can define the operator

$$\Gamma_S : L^2(\mathbb{R}, \mathcal{H}) \to W^{1,2}(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{W})$$

by

$$\Gamma_s v(s) := \int_{\mathbb{R}} G(s-\tau)v(\tau)d\tau = \int_{\mathbb{R}} G(-s)v(s+\tau)d\tau.$$

This is well defined due to the following estimate:

$$\begin{aligned}\int_{\mathbb{R}} \|G(-s)v(s+\tau)\|_{L^2(\mathbb{R}\times S^1)} d\tau &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|G(-\tau)v(s+\tau)\|^2_{L^2(S^1)} ds\right)^{1/2} d\tau \\ &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-2\kappa|\tau|} \|v(s+\tau)\|^2_{L^2(S^1)} ds\right)^{1/2} d\tau \\ &= \int_{\mathbb{R}} e^{-2\kappa|\tau|} \|v\|_{L^2(\mathbb{R}\times S^1)} d\tau < +\infty.\end{aligned}$$

We claim that Γ_S is the inverse of D_S. Indeed, if $v \in L^2(\mathbb{R}, \mathcal{H})$ and $\Gamma_S v = w$ then write

$$w = w^+ + w^-,$$

where

$$w^+(s) = \int_{-\infty}^{s} E_{-A^+}(s-\tau)P^+v(\tau)d\tau, \quad w^-(s) = -\int_{s}^{+\infty} E_{A^-}(\tau - s)P^-v(\tau)d\tau.$$

Then

$$\frac{d}{ds}w^+(s) = P^+v(s) - A^+\int_{-\infty}^{s} E_{-A^+}(s-\tau)P^+v(\tau)d\tau = P^+v(s) - A^+w^+(s),$$

and similarly

$$\frac{d}{ds}w^-(s) = P^-v(s) - A^-w^-(s),$$

and hene $\frac{dw}{ds} = v - Aw$, so that $D_S\Gamma_S v = v$. Similarly $\Gamma_S D_S u = u$ for every $u \in W^{1,2}(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{W})$. Thus D_S is invertible. This proves Theorem 5.7 for the case $p = 2$. We now prove the general case.

Proof. (of Theorem 5.7).
Suppose $p > 2$. We first show there exists a constant $c > 0$ such that

$$\|u\|_{W^{1,p}([k,k+1]\times S^1)} \leq c\left(\|D_S u\|_{L^p([k-1,k+2]\times S^1)} + \|u\|_{L^2([k-1,k+2]\times S^1)}\right). \tag{5.3}$$

To prove (5.3) we first note that there exists a constant $c_1 > 0$ such that

$$\|u\|_{W^{1,p}([k,k+1]\times S^1)} \leq c\left(\|D_S u\|_{L^p([k-1/2,k+3/2]\times S^1)} + \|u\|_{L^2([k-1,/2k+3/2]\times S^1)}\right)$$

(see the proof of part (2) of Theorem (4.12)). Since $W^{1,2}$ embeds continuously into L^p for every $p \in [1, +\infty)$, there exists $c_2 > 0$ such that

$$\|u\|_{L^p([k-1/2,k+3/2]\times S^1)} \leq c_2\left(\|u\|_{W^{1,2}([k-1/2,k+3/2]\times S^1)}\right).$$

Then the first estimate but for $p = 2$ gives a constant $c_3 > 0$ such that

$$\begin{aligned}\|u\|_{W^{1,2}([k-1/2,k+3/2]\times S^1)} &\le c_3\left(\left\|\|D_S u\|_{L^2([k-1,k+2]\times S^1)} + \|u\|_{L^2([k-1,k+2]\times S^1)}\right\|\right)\\ &\le c_4\left(\left\|\|D_S u\|_{L^p([k-1,k+2]\times S^1)} + \|u\|_{L^2([k-1,k+2]\times S^1)}\right\|\right),\end{aligned}$$

since $p > 2$. Putting this together we deduce an estimate of the form (5.3).

Next, we claim that there exist constants $a, b > 0$ such that if $u \in W^{1,2}$ and $D_S u \in L^p$ then $u \in W^{1,p}$ and

$$\|u\|_{L^p(\mathbb{R},\mathcal{H})} \le a\,\|D_S u\|_{L^p(\mathbb{R}\times S^1)}, \tag{5.4}$$

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \le b\left(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p(\mathbb{R},\mathcal{H})}\right). \tag{5.5}$$

To see this we first note that by assumption $v := D_S u$ belongs to $L^2(\mathbb{R},\mathcal{H})$ and satisfies

$$\|v\|_{L^p(\mathbb{R},\mathcal{H})} = \left(\int_{\mathbb{R}} \|v\|^p_{\mathcal{H}}\,ds\right)^{1/2} = \left(\int_{\mathbb{R}}\left(\int_{S^1}|v|^2\,dt\right)^{p/2} ds\right)^{1/2} \le \|v\|_{L^p(\mathbb{R}\times S^1)},$$

and hence v also belongs to $L^p(\mathbb{R},\mathcal{H})$. Since $u \in W^{1,2}$, from above we know that $u = \Gamma_S v$, and hence by Young's inequality and (5.2) we obtain

$$\|u\|_{L^p(\mathbb{R},\mathcal{H})} = \|\Gamma_S v\|_{L^p(\mathbb{R},\mathcal{H})} = \|G * v\|_{L^p(\mathbb{R},\mathcal{H})} \le \|G\|_{L^1(\mathbb{R},\mathsf{L}(\mathcal{H}))}\|v\|_{L^p(\mathbb{R},\mathcal{H})} \le \frac{2}{\kappa}\|v\|_{L^p(\mathbb{R},\mathcal{H})}.$$

Combined with the previous inequality, this proves (5.4). Now to prove (5.5), since $W^{1,2}$ embeds into L^p_{loc}, both u and $\bar{\partial}u = D_S u - Su$ belong to L^p_{loc}. Thus by elliptic regularity, $u \in W^{1,p}_{\mathrm{loc}}$. Using (5.3) we see that

$$\begin{aligned}\|u\|_{L^{1,p}([k,k+1]\times S^1)} &\le c\left(\int_{k-1}^{k+1}\int_{S^1}|D_S u|^p\,dtds + \left(\int_{k-1}^{k+2}\int_{S^1}|u|^2\,dsds\right)^{p/2}\right)\\ &\le c_1\int_{k-1}^{k+2}\left(\int_{S^1}|D_S u|^p\,dt + \left(\int_{S^1}|u|^2\,dt\right)^{p/2}\right)ds.\end{aligned}$$

Summing over k we obtain (5.5).

Combining (5.4) and (5.5) together we see that if $u \in C^\infty_c$ then there exists a constant $c > 0$ such that

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \le c\,\|D_S u\|_{L^p(\mathbb{R}\times S^1)},$$

and by density this actually holds for all $u \in W^{1,p}$. Thus the map $D_S \in \mathsf{L}(W^{1,p}, L^p)$ is injective and has closed range. If $v \in L^p \cap L^2$ then since we already know the result for $p = 2$ there exists $u \in W^{1,p}$ such that $D_S u = v$. But then from the above we know that this implies $u \in W^{1,p}$. Thus $D_S(W^{1,p})$ contains the dense subspace $L^p \cap L^2$. Since $D_S(W^{1,p})$ is closed in L^p we see that $D_S(W^{1,p}) = L^p$, and thus D_S is also surjective. ■

Exercise 5.9. Use a duality argument to complete the proof of Theorem 5.7 in the case where $1 < p < 2$. *Hint:* See [AD10, Proposition 8.7.15] if you get stuck.

We now move onto proving the main result of this section, Theorem 5.6. In order to do so we first recall some more standard facts about Fredholm operators.

Let X and Y denote real Banach spaces. Let $\mathsf{L}(X,Y)$ denote the set of continuous linear maps and $\mathsf{L}_c(X,Y)$ the subspace of compact operators. We say that $T \in \mathsf{L}(X,Y)$

is *semi-Fredholm* if $\operatorname{ran} T$ is a closed subspace of Y and at least one of ker T and $\operatorname{coker} T$ are finite dimensional. If T is semi-Fredholm then it still makes sense to define the index of T to be

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T \in \mathbb{Z} \cup \{\pm\infty\}.$$

Thus a Fredholm operator is precisely a semi-Fredholm with finite Fredholm index. We will need the following result, which can be found in [Kat76, Section IV.5].

Proposition 5.10. *The set of semi-Fredholm operators is open in* $\mathsf{L}(X,Y)$, *and the index is a continuous function. If* T *is semi-Fredholm and* $K \in \mathsf{L}_c(X,Y)$ *then* $T+K$ *is semi-Fredholm of the same index.*

We will also need the following simple lemma.

Lemma 5.11. *Suppose* X, Y, Z *are Banach spaces and* $T \in \mathsf{L}(X,Y)$ *and* $K \in \mathsf{L}(X,Z)$. *If there exists a number* $c > 0$ *such that*

$$\|x\|_X \le c\left(\|Tx\|_Y + \|Kx\|_Z\right), \quad \text{for all } x \in X, \tag{5.6}$$

then T *has finite dimensional kernel and closed range, and hence is a semi-Fredholm operator with index* $\operatorname{ind} T \in \mathbb{Z} \cup \{-\infty\}$.

Proof. We show that $\ker T \cap B_X$ is compact, where B_X is the ball of radius 1 in X. This implies ker T is finite dimensional. If $(x_k) \subset \ker T \cap B_X$ then since K is compact, up to a subsequence we may assume (Kx_k) is convergent, and hence Cauchy. Feeding this into (5.6) we see that

$$\|x_h - x_k\|_X \le c\,\|Kx_h - Kx_k\|_Z \to 0.$$

Thus (x_k) is Cauchy in X, and hence converges up to a subsequence.

It remains to show that $\operatorname{ran} T$ is closed. Since ker T is finite dimensional, by the Hahn-Banach theorem there exists a topological complement $X_0 \subset X$ of ker T (i.e. X_0 is a closed subspace such that $X = X_0 \oplus \ker T$, $X_0 \cap \ker T = (0)$). Suppose $y \in \overline{\operatorname{ran} T}$. Then by definition there exists $(x_k) \subset X_0$ such that $Tx_k \to y$. We claim that $\|x_k\|_X$ is bounded. If not then passing to a subsequence we may assume that $\|x_k\|_X \to +\infty$. Set $x'_k := x_k / \|x_k\|_X \in X_0 \cap B_X$. Then $Tx'_k \to 0$ and Kx'_k converges in Z (up to another subsequence). Feeding this into (5.6) again, we see that

$$\left\|x'_h - x'_k\right\|_X \le c\left(\left\|Tx'_h - Tx'_k\right\|_Y + \left\|Kx'_h - Kx'_k\right\|_Z\right) \to 0.$$

Thus $x'_k \to x' \in X_0 \cap B_X$ (since X_0 is closed). But also $x' \in \ker T$, contradicting $X_0 \cap \ker T = (0)$.

Thus (x_k) is bounded. As before, this implies that up to a subsequence (Kx_k) is convergent, and hence feeding this one more time into (5.6) shows that (x_k) is Cauchy. Thus $x_k \to x$ with $Tx = y$. Thus $y \in \operatorname{ran} T$ and so $\operatorname{ran} T$ is closed as claimed. ∎

Let us now prove Theorem 5.6.

Proof. Given $\sigma \ge 0$ define

$$S^+_\sigma(s,t) := \begin{cases} S(s,t), & s \ge \sigma, \\ S(\sigma,t), & s < \sigma, \end{cases}$$

and

$$S^-_\sigma(s,t) := \begin{cases} S(-\sigma,t), & s > -\sigma, \\ S(s,t), & s \le -\sigma. \end{cases}$$

Then

$$\left\| D_{S_\sigma^\pm} - D_{S^\pm} \right\|_{\mathsf{L}(W^{1,p},L^p)} \leq \left\| S_\sigma^\pm - S^\pm \right\|_{L^\infty} \to 0,$$

and hence as the set of invertible linear operators is open in $\mathsf{L}(W^{1,p}, L^p)$, there exists $\sigma > 0$ such that $D_{S_\sigma^\pm}$ is invertible. Thus there exists $c_1 > 0$ such that

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c_1 \left\| D_{S_\sigma^\pm} u \right\|_{L^p(\mathbb{R}\times S^1)}.$$

Thus if $u \in W^{1,p}(\mathbb{R} \times S^1)$ is supported in $(\mathbb{R}\backslash[-\sigma - 1, \sigma + 1]) \times S^1$ then

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c_1 \|D_S u\|_{L^p(\mathbb{R}\times S^1)}. \tag{5.7}$$

Now select a smooth function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(s) = 1$ for $|s| \leq \sigma+1$ and $\beta(s) = 0$ for $|s| \geq \sigma + 2$. As in the proof of Theorem 4.12, we write $u = \beta u + (1 - \beta)u$ and estimate:

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq \|\beta u\|_{W^{1,p}(\mathbb{R}\times S^1)} + \|(1 - \beta)u\|_{W^{1,p}(\mathbb{R}\times S^1)}.$$

We estimate the first term using (4.7) to obtain:

$$\begin{aligned} \|\beta u\|_{W^{1,p}(\mathbb{R}\times S^1)} &\leq c_2 \left(\|D_S(\beta u)\|_{L^p(\mathbb{R}\times S^1)} + \|\beta u\|_{L^p(\mathbb{R}\times S^1)} \right) \\ &\leq c_3 \left(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p([-\sigma-2,\sigma+2]\times S^1)} \right). \end{aligned}$$

Using (5.7) we estimate the second term by

$$\begin{aligned} \|(1 - \beta)u\|_{W^{1,p}(\mathbb{R}\times S^1)} &\leq c_1 \|D_S(1 - \beta)u\|_{L^p(\mathbb{R}\times S^1)} \\ &= c_1 \left\|D_S u - \beta' u\right\|_{L^p(\mathbb{R}\times S^1)} \\ &\leq c \|D_S u\|_{L^p(\mathbb{R}\times S^1)} + c_4 \|u\|_{L^p([-\sigma-2,\sigma+2]\times S^1)}. \end{aligned}$$

Combining these two estimates we see that there exists a constant $c_0 > 0$ such that

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c_0 \left(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p([-\sigma-2,\sigma+2]\times S^1)} \right), \quad \text{for all } u \in W^{1,p}(\mathbb{R} \times S^1).$$

Now we apply Lemma 5.11 with $X = W^{1,p}(\mathbb{R} \times S^1)$, $Y = L^p(\mathbb{R} \times S^1)$ and $Z = L^p([-\sigma - 2, \sigma + 2] \times S^1)$, with $T = D_S$ and K the restriction operator

$$W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p([-\sigma - 2, \sigma + 2] \times S^1, \mathbb{R}^{2n}).$$

By Rellich's compactness theorem, K is indeed a compact operator. Thus Lemma 5.11 implies that D_S has finite dimensional kernel and closed range. To complete the proof we must show that the range of D_S has finite codimension in L^p. Equivalently, letting $q > 1$ be such that $1/p + 1/q = 1$, we must show that the annhilator

$$(\operatorname{ran} D_S)^\circ := \left\{ v \in L^p(\mathbb{R} \times S^1) \mid \int_{\mathbb{R}\times S^1} D_S u \cdot v ds dt = 0, \text{ for all } u \in W^{1,p}(\mathbb{R} \times S^1) \right\}$$

is finite dimensional. But if $v \in (\operatorname{ran} D_S)^\circ$ then v is a weak solution to the equation $\overline{\partial} v = S^* v$, and hence by Theorem 4.12, v is actually a strong solution. Thus

$$(\operatorname{ran} D_S)^\circ \subset \ker D_{-S^*}.$$

But the operator D_{-S^*} is of the same form as D_S (i.e. $-S^*$ has the same properties as D_S). So from what we already know, $\ker D_{-S^*}$ is finite dimensional. Thus the same is true of $(\operatorname{ran} D_S)^\circ$. This completes the proof of Theorem 5.6. ■

Exercise 5.12. Show that actually one has equality

$$(\operatorname{ran} D_S)^\circ = \ker\, D_{-S^*}.$$

The aim now is to compute the Fredholm index of the operator $D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. This will take some time, and will not be completed until the next section. Firstly, we note that in order to compute the index it suffices to work with in the case $p = 2$.

Lemma 5.13. *The index of D_S is independent of the choice of $1 < p < +\infty$.*

Proof. We know from elliptic regularity that if $u \in \ker\, D_S$ then u is smooth. In particular, $\dim \ker\, D_S$ does not depend on the choice of p. Moreover from Exercise 5.12 we know that $\operatorname{coker} D_S \cong \ker\, D_{-S^*}$, and hence $\dim \operatorname{coker} D_S = \dim \ker\, D_{-S^*}$ also does not depend on p. ■

Next, we prove that the index of D_S only depends on the asymptotes $S^\pm(t) := \lim_{s \to \pm\infty} S(s, \cdot)$, which are by assumption symmetric matrices such that the fundamental solutions $\Psi^\pm$ from (??) are such that $\det(\Psi^\pm(1) - \mathbb{1}) \neq 0$. Fix two paths $S^\pm : S^1 \to \mathsf{Sym}(\mathbb{R}^{2n})$ with the property that if $\Psi^\pm : [0,1] \to \mathsf{Sp}(\mathbb{R}^{2n}, \omega_0)$ are defined by

$$(\Psi^\pm)'(t) = J_0 S(t) \Psi(t), \quad \Psi^\pm(0) = \mathbb{1},$$

then

$$\det(\Psi^\pm(1) - \mathbb{1}) \neq 0.$$

Let $\mathcal{F}(S^-, S^+)$ denote the set of all operators $D_S : W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ of the form

$$D_S = \partial_s + J_0 \partial_t + S(s,t),$$

where $S \in C^\infty(\mathbb{R} \times S^1, \mathsf{L}(\mathbb{R}^{2n})) \cap L^\infty(\mathbb{R} \times S^1, \mathsf{L}(\mathbb{R}^{2n}))$ is a smooth map such that $\lim_{s \to \pm\infty} S(s,t) = S^\pm(t)$, uniformly in t. Thus every $D_S \in \mathcal{F}(S^-, S^+)$ is Fredholm.

Proposition 5.14. *The space $\mathcal{F}(S^-, S^+)$ is contractible. Thus the index of D_S is constant on $\mathcal{F}(S^-, S^+)$:*

$$\operatorname{ind} D_{S_1} = \operatorname{ind} D_{S_2}, \quad \text{for all } D_{S_1}, D_{S_2} \in \mathcal{F}(S^-, S^+).$$

Proof. Fix $D_{S_0} \in \mathcal{F}(S^-, S^+)$. Define a map $\Theta : [0,1] \times \mathcal{F}(S^-, S^+) \to \mathcal{F}(S^-, S^+)$ by

$$\Theta(r, D_S) := \partial_s + J_0 \partial_t + \tau S_0 + (1 - \tau) S.$$

Thus $\Theta(0, \cdot) = \mathbb{1}$ and $\Theta(1, D_S) = D_{S_0}$ for all $D_S \in \mathcal{F}(S^-, S^+)$. To complete the proof we must show that Θ is continuous. Let $(u_k) \in W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ with $\|u_k\|_{W^{1,2}(\mathbb{R} \times S^1)} = 1$. Suppose $r_k \to r$ and $D_{S_k} \to D_S$ in the space $\mathsf{L}(W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}), L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n}))$. We must show that

$$\|\Theta(r_k, D_{S_k}) u_k - \Theta(r, D_S) u_k\|_{L^2(\mathbb{R} \times S^1)} \to 0.$$

Since $D_{S_k} \to D_S$ one has $\|S_k - S\|^{\text{op}} \to 0$ and thus

$$\begin{aligned} \|\Theta(r_k, D_{S_k}) u_k - \Theta(r, D_S) u_k\|_{L^2(\mathbb{R} \times S^1)} &= \|((1 - r_k) S_k + r_k S_0 - (1 - r) S_k + r S)\, u_k\|_{L^2(\mathbb{R} \times S^1)} \\ &\leq 2 \|S_k - S\|^{\text{op}} \|u_k\|_{W^{1,2}(\mathbb{R} \times S^1)} \to 0. \end{aligned}$$

■

The main result of this section is the following result, which is due to Floer and Hofer [FH93, Proposition 9] and Schwarz [Sch95, Theorem 3.2.12], and states that the Fredholm index is additive.

Theorem 5.15 (Additivity of index)**.** *Suppose S^-, S^0, S^+ are three paths $S^1 \to \mathsf{Sym}(\mathbb{R}^{2n})$ whose fundamental solutions $\Psi^-, \Psi^0, \Psi^+ : [0,1] \to \mathsf{Sp}(\mathbb{R}^{2n}, \omega_0)$ are all non-degenerate. Suppose $D_R \in \mathcal{F}(S^-, S^0)$,$D_S \in \mathcal{F}(S^0, S^+)$ and $D_T \in \mathcal{F}(S^-, S^+)$. Then*

$$\operatorname{ind} D_T = \operatorname{ind} D_R + \operatorname{ind} D_S.$$

This will take quite a while to prove.

Definition 5.16. We say that $D_S \in \mathcal{F}(S^-, S^+)$ is *asymptotically constant* if there exists $s_0 \geq 0$ such that

$$S(s,t) = \begin{cases} S^-(t), & s \geq s_0, \\ S^+(t), & s \leq -s_0. \end{cases}$$

Given $D_S \in \mathcal{F}(S^-, S^+)$ and a real number $\rho \in \mathbb{R}$, we define $D_{S_\rho} \in \mathcal{F}(S^-, S^+)$ by setting

$$S_\rho(s,t) := S(s+\rho, t).$$

Note that if D_S is asymptotically constant then so is D_{S_ρ} for all $\rho \in \mathbb{R}$. We now introduce a 'gluing' operation.

Definition 5.17. Suppose S^-, S^0, S^+ are three paths $S^1 \to \mathsf{Sym}(\mathbb{R}^{2n})$ whose fundamental solutions $\Psi^-, \Psi^0, \Psi^+ : [0,1] \to \mathsf{Sp}(\mathbb{R}^{2n}, \omega_0)$ are all non-degenerate. Suppose $D_R \in \mathcal{F}(S^-, S^0)$ and $D_S \in \mathcal{F}(S^0, S^+)$ are both asymptotically constant. Choose $\rho_0 > 0$ large enough so that

$$R_{\rho_0}(s,t) = S^0(t), \quad \text{for all } (s,t) \in [-1, +\infty) \times S^1,$$

$$S_{-\rho_0}(s,t) = S^0(t), \quad \text{for all } (s,t) \in (-\infty, 1] \times S^1.$$

Then for $\rho \geq \rho_0$ there is a well defined asymptotically constant operator $D_{T_\rho} \in \mathcal{F}(S^-, S^+)$ defined by $D_{T_\rho} = \overline{\partial} + T_\rho$, where

$$T_\rho(s,t) = \begin{cases} R_\rho(s,t), & (s,t) \in (-\infty, 0] \times S^1, \\ S_{-\rho}(s,t), & (s,t) \in [0, +\infty) \times S^1. \end{cases}$$

In particular,

$$T_\rho(s,t) = S^0(t), \quad \text{for all } (s,t) \in [-\rho, \rho] \times S^1. \tag{5.8}$$

Theorem 5.15 follows from the next result and Proposition 5.14

Theorem 5.18. *One has*

$$\operatorname{ind} D_{T_\rho} = \operatorname{ind} D_R + \operatorname{ind} D_S.$$

The proof will take some time. We now introduce a *stabilisation* trick which allows us to reduce to the case of surjective operators.

Definition 5.19. Suppose $F : \mathbb{R}^p \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is a smooth linear map. Define $D_S^F : W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times \mathbb{R}^p \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ by

$$D_S^F(u, x) := D_S u + F(x).$$

Note that if $e_i : i = 1, \ldots, p$ and $e'_j : j = 1, \ldots, q$ are the standard bases of $\mathbb{R}^p$ and $\mathbb{R}^q$ then F is equivalent to a p-tuple $(f_1, \ldots, f_p)$ of functions belonging to $L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$:

$$F(x) = \sum_{i=1}^{p} x_i f_i, \quad x = \sum_{i=1}^{p} x_i e_i.$$

In particular, if we choose the f_i such that coker D_S is contained in the space of the f_i then the operator D_S^F is surjective. This can be done whenever $\dim \operatorname{coker} D_S \leq p$. Moreover, since the set of surjective operators is open, whenever $\dim \operatorname{coker} D_S \leq p$ we can always choose $F : \mathbb{R}^p \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ with the property that

$$\operatorname{supp}(F(x)) \subset [-R, R] \times S^1, \quad \text{for all } x \in \mathbb{R}^p. \tag{5.9}$$

Exercise 5.20. Prove this last assertion.

Definition 5.21. Given $u \in L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, and $\rho \in \mathbb{R}$, we define

$$u_\rho(s,t) := u(s+\rho, t).$$

We now define an operator

$$L_\rho := D_{T_\rho}^{F_\rho + F_{-\rho}} : W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times \mathbb{R}^{2p} \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

given by

$$L_\rho(u, x, y) := D_{T_\rho} u + F(x)_\rho + F(y)_{-\rho}.$$

Thus

$$L_\rho(u,x,y)(s,t) = \begin{cases} \bar\partial u(s,t) + R(s+\rho,t)u(s,t) + F(x)(s+\rho,t) + F(y)(s-\rho,t), & s \leq 0, \\ \bar\partial u(s,t) + S(s-\rho,t)u(s,t) + F(x)(s+\rho,t) + F(y)(s-\rho,t) & s \geq 0. \end{cases}$$

The main step in the forthcoming proof is the following statement:

Proposition 5.22. *Suppose D_R^F and D_S^F are both surjective and that F satisfies (5.9). Then for all $\rho \gg 0$, the operator L_ρ is also surjective. Moreover for all $\rho \gg 0$, if π_ρ denotes the L^2-projection onto* ker L_ρ *then the map*

$$\phi_\rho : \ker D_R^F \times \ker D_S^F \to \ker L_\rho,$$

$$\phi_\rho((u,x),(v,y)) := \pi_\rho((u_\rho + v_{-\rho}, x, y)) \tag{5.10}$$

is an isomorphism.

Before proving Proposition 5.22, let us see how this result implies Theorem 5.18.

Proof. *(Of Theorem 5.18)*
Consider the exact sequence of finite dimensional vector spaces:

$$0 \to \ker D_R \xrightarrow{a} \ker D_R^F \xrightarrow{b} \mathbb{R}^p \xrightarrow{c} \operatorname{coker} D_R \to 0,$$

where $a(u) := (u, 0)$, $b(u,x) := x$ and $c(x) := D_R^F(0,x) + \operatorname{ran} D_R$. It is clear that a is injective, $\ker b = \operatorname{im} a$ and $\ker c = \operatorname{im} b$, and finally c is surjective as by assumption D_R^F is surjective. Exactness implies that

$$\dim \ker D_R + p = \dim \ker D_R^F + \dim \operatorname{coker} D_R,$$

and hence we deduce that

$$\operatorname{ind} D_R = \dim \ker D_R^F - p.$$

Applying the same logic to D_{T_ρ} and D_S (since by assumption they are both surjective), we obtain

$$\begin{aligned}\operatorname{ind} D_{T_\rho} &= \dim \ker L_\rho - 2p \\ &\overset{(*)}{=} \dim \ker D_R^F + \dim \ker D_S^F - 2p \\ &= \operatorname{ind} D_R + \operatorname{ind} D_S,\end{aligned}$$

where $(*)$ used the fact that ϕ_ρ is an isomorphism. ■

We now move on to proving Proposition 5.22. In fact, we will prove only half of the result: we will proof that ϕ_ρ is surjective for all $\rho \gg 0$.

Proof. (that the map ϕ_ρ from (5.10) is surjective for all $\rho \gg 0$)
It suffices to show that there exists a constant $\rho_1 \geq \rho_0 + 1 + R$ and a constant $c > 0$ such that for all $\rho \geq \rho_1$ one has

$$\|L_\rho(u, x, y)\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^{2p}} \geq c\left(\|u\|_{L^2(\mathbb{R}\times S^1)} + |x|^2 + |y|^2\right)^{1/2} \tag{5.11}$$

for all $(u, x, y) \in \mathcal{R}_\rho$, where $\mathcal{R}_\rho$ denotes the set of tuples $(u, x, y) \in W^{1,2}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \times \mathbb{R}^{2p}$ such that

$$\langle\langle u, v_\rho + w_{-\rho}\rangle\rangle_{L^2(\mathbb{R}\times S^1)} + \langle x, a\rangle + \langle y, b\rangle = 0, \text{ for all } (v, a) \in \ker D_R, \text{ and } (w, b) \in \ker D_S.$$

If (5.11) is false then we can find sequences $(u_k, x_k, y_k) \in W^{1,2}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \times \mathbb{R}^{2p}$ such that

$$\|u_k\|_{L^2(\mathbb{R}\times S^1)} + |x_k|^2 + |y_k|^2 = 1, \tag{5.12}$$

and a sequence $\rho_k \to +\infty$ such that

$$(u_k, x_k, y_k) \in \mathcal{R}_{\rho_k},$$

$$\|L_{\rho_k}(u_k, x_k, y_k)\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^{2p}} \to 0.$$

Define a smooth map $\vartheta : \mathbb{R} \to [0, 1]$ such that $\vartheta(s) = 1$ for $|s| \leq 1/2$ and $\vartheta(s) = 0$ for $|s| \geq 1$. Given $r > 0$ let $\vartheta_r(s) := \vartheta(s/r)$. Choose $r_k > 0$ such that

$$\tfrac{1}{2}\rho_k < r_k < \tfrac{3}{4}\rho_k, \tag{5.13}$$

and set

$$f_k := \vartheta_{r_k} u_k.$$

We claim that $\|f_k\|_{W^{1,2}(\mathbb{R}\times S^1)} \to 0$. Here the key point is that since the operator S^0 is non-degenerate in the sense that $\Psi^0(1)$ does not have 1 as an eigenvalue, Theorem 5.7 implies that $D_{S^0} : W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is invertible. Thus it suffices to show that $\|D_{S^0} f_k\|_{L^2(\mathbb{R}\times S^1)} \to 0$. Using (5.8), (5.12) and (5.13) we see that for k large enough,

$$\begin{aligned}\|D_{S_0} f_k\|_{L^2(\mathbb{R}\times S^1)} &\leq \tfrac{1}{r_k}\left\|\vartheta'(\cdot)u_k\right\|_{L^2(\mathbb{R}\times S^1)} + \|\vartheta(\cdot/r_k)D_{S_0}u_k\|_{L^2(\mathbb{R}\times S^1)} \\ &\leq \tfrac{c}{r_k} + \|L_{\rho_k}(u_k, x_k, y_k)\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^{2p}} \to 0,\end{aligned}$$

where the last line used the fact that $F(x_k)_{\rho_k}|_{L^2((-r_k,r_k)\times S^1)} = F(y_k)_{-\rho_k}|_{L^2((-r_k,r_k)\times S^1)} = 0$ for k large enough due to (5.9). We conclude that

$$\|u_k\|_{W^{1,2}((-\rho_k/4,\rho_k/4)\times S^1)} \to 0. \tag{5.14}$$

Now choose a smooth function $\beta^- : \mathbb{R} \to [0,1]$ such that $\beta^-(s) = 1$ for $s \le -1$ and $\beta^-(s) = 0$ for $s \ge 0$. Abbreviate

$$v_k(s,t) := \beta^-(s-\rho_k)u_k(s-\rho_k,t).$$

We claim that (v_k, x_k) converges to some $(v,x) \in \ker D_R^F$. For this we compute

$$\begin{aligned}\left\|D_R^F(v_k,x_k)\right\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^p} &= \left\|L_{\rho_k}(\beta^- u_k, x_k, 0)\right\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^{2p}} \\ &= \left\|(\beta^-)'u_k + \beta^- D_{T_{\rho_k}} u_k + F(x_k)_{\rho_k}\right\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^{2p}} \\ &\le c\,\|u_k\|_{L^2((-1,1)\times S^1)} + \|L_{\rho_k}(u_k,x_k,y_k)\|_{L^2(\mathbb{R}\times S^1)\times\mathbb{R}^{2p}} \to 0,\end{aligned}$$

where the last line used (5.14). Since (v_k, x_k) is bounded in $W^{1,2}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \times \mathbb{R}^p$ we see that (up to a subsequence) $(v_k, x_k) \to (v, x) \in \ker D_R^F$ as claimed. Similarly by setting $\beta^+(s) := \beta^-(-s)$ and setting

$$w_k(s,t) := \beta^+(s+\rho_k)u_k(s+\rho_k,t)$$

we see that $(w_k, y_k) \to (w, y) \in \ker D_S^F$. From (5.12) we see that

$$\|v\|_{L^2(\mathbb{R}\times S^1)} + \|w\|_{L^2(\mathbb{R}\times S^1)} + |x|^2 + |y|^2 = 1.$$

But now we obtain the desired contradiction by observing:

$$\begin{aligned}1 &= \|v\|_{L^2(\mathbb{R}\times S^1)} + \|w\|_{L^2(\mathbb{R}\times S^1)} + |x|^2 + |y|^2 \\ &= \lim_{k\to+\infty}\Big(\langle\langle\beta^- u_k, v_{\rho_k}\rangle\rangle_{L^2(\mathbb{R}\times S^1)} + \langle\langle\beta^+ u_k, w_{-\rho_k}\rangle\rangle_{L^2(\mathbb{R}\times S^1)} + \langle x, x_k\rangle + \langle y, y_k\rangle\Big) \\ &= \lim_{k\to+\infty}\Big(\langle\langle u_k, v_{\rho_k} + w_{-\rho_k}\rangle\rangle_{L^2(\mathbb{R}\times S^1)} + \langle x, x_k\rangle + \langle y, y_k\rangle\Big) \\ &= \lim_{k\to+\infty} 0 = 0,\end{aligned}$$

since by assumption $(u_k, x_k, y_k) \in \mathcal{R}_{\rho_k}$. ■

Exercise 5.23. Complete the proof of Proposition 5.22 by showing that L_ρ is surjective and ϕ_ρ is injective for all $\rho \gg 0$. *Hint:* This is hard! See [Sch95, Section 3.2] for a detailed proof.

CHAPTER 6

Index computations

We now recall some properties of the symplectic linear group that we will need later. Recall that

$$\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)=\left\{W\in\mathsf{L}(\mathbb{R}^{2n})\mid W^*J_0W=J_0\right\}.$$

This is a Lie subgroup of $\mathrm{GL}(\mathbb{R}^{2n})$ and the Lie algebra is

$$\mathfrak{sp}(\mathbb{R}^{2n},\omega_0)=\left\{A\in\mathsf{L}(\mathbb{R}^{2n})\mid A^*J_0+J_0A=0\right\}.$$

Equivalently, $A\in\mathfrak{sp}(\mathbb{R}^{2n},\omega_0)$ if and only if $A=J_0S$ for some symmetric matric $S\in\mathrm{Sym}(\mathbb{R}^{2n})$. In particular, any continuously differentiable path $\Psi:[0,1]\to\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)$ satisfying $\Psi(0)=\mathbb{1}$ can be (possibly after reparametrizing) written uniquely as

$$\Psi(t)=\exp\left(J_0\int_0^t S(s)ds\right)\tag{6.1}$$

for some path $S:S^1\to\mathrm{Sym}(\mathbb{R}^{2n})$, where $\exp:\mathfrak{sp}(\mathbb{R}^{2n},\omega_0)\to\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)$ is the exponential map. Note that then Ψ satisfies

$$\Psi'(t)=J_0S(t)\Psi(t).$$

Elementary linear algebra shows that any $W\in\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)$ can be written as

$$W=PO,\tag{6.2}$$

where P is the positive symmetric symplectic matrix $P=(WW^*)^{1/2}$ and O is the orthogonal symplectic matrix $O=(WW^*)^{-1/2}W$. This shows that

$$\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)\cong\left(\mathrm{Sym}^+(\mathbb{R}^{2n})\cap\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)\right)\times\left(O(\mathbb{R}^{2n})\cap\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)\right).$$

Exercise 6.1. Show that $\mathrm{Sym}^+(\mathbb{R}^{2n})\cap\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)$ is homeomorphic to an $n(n+1)$ dimensional vector space. Show that if $O\in O(\mathbb{R}^{2n})\cap\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)$ then

$$O=\begin{pmatrix}X&-Y\\Y&X\end{pmatrix}\tag{6.3}$$

for some pair $X,Y\in\mathsf{L}(\mathbb{R}^{2n})$, and that the map

$$O\mapsto X+iY$$

defines a homeomorphism

$$O(\mathbb{R}^{2n})\cap\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)\cong\mathrm{U}(\mathbb{C}^n).$$

Recall that $\pi_1(\mathrm{U(n)})\cong\mathbb{Z}$, and the map $\det:\mathrm{U}(\mathbb{C}^n)\to S^1$ induces an isomorphism $\det_*:\pi_1(\mathrm{U}(\mathbb{C}^n))\to\mathbb{Z}$.

Definition 6.2. Define $\widetilde{\det}:\mathrm{Sp}(\mathbb{R}^{2n},\omega_0)\to S^1$ by setting

$$\widetilde{\det}(W):=\det(X+iY),$$

where $W=PO$ as in (6.2) and $O=\begin{pmatrix}X&-Y\\Y&X\end{pmatrix}$ as in (6.3).

We can now define an integer-valued function on the set of loops $\Phi : S^1 \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

Definition 6.3. Define the *Maslov index* of a loop $\Phi : S^1 \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$ by setting

$$\mu(\Phi) := \text{degree}\left(t \mapsto \widetilde{\det}(\Phi(t))\right) \in \mathbb{Z}.$$

We now move on to defining the Conley-Zehnder index. Firstly let us write

$$\mathrm{Sp}^+(\mathbb{R}^{2n}, \omega_0) := \left\{W \in \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0) \mid \det(\mathbb{1} - \Psi(1)) > 0\right\},$$
$$\mathrm{Sp}^0(\mathbb{R}^{2n}, \omega_0) := \left\{W \in \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0) \mid \det(\mathbb{1} - \Psi(1)) = 0\right\},$$
$$\mathrm{Sp}^-(\mathbb{R}^{2n}, \omega_0) := \left\{W \in \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0) \mid \det(\mathbb{1} - \Psi(1)) < 0\right\},$$

so that

$$\mathrm{Sp}(\mathbb{R}^{2n}, \omega_0) = \mathrm{Sp}^-(\mathbb{R}^{2n}, \omega_0) \sqcup \mathrm{Sp}^0(\mathbb{R}^{2n}, \omega_0) \sqcup \mathrm{Sp}^-(\mathbb{R}^{2n}, \omega_0).$$

Exercise 6.4. Show that $\mathrm{Sp}^{\pm}(\mathbb{R}^{2n}, \omega_0)$ are both path connected, and that any closed loop contained in either $\mathrm{Sp}^+(\mathbb{R}^{2n}, \omega_0)$ or $\mathrm{Sp}^-(\mathbb{R}^{2n}, \omega_0)$ is contractible. *Hint:* Look at [SZ92, Lemma 3.2] if you get stuck.

Definition 6.5. Let $\mathcal{S}^*(2n) \subset C^0([0,1], \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0))$ denote the set of continuous paths satisfying

$$\Psi(0) = \mathbb{1}, \quad \Psi(1) \notin \mathrm{Sp}^0(\mathbb{R}^{2n}, \omega_0),$$

endowed with the compact-open topology.

Our goal in what follows is to find a way to classify the connected components of $\mathcal{S}^*(2n)$. This is useful because of the following exercise.

Exercise 6.6. Improve Proposition 5.14 to show that the index of an operator $D_S \in \mathcal{F}(S^-, S^+)$ only depends on the connected components of $\mathcal{S}^*(2n)$ that the paths $\Psi^{\pm}$ belong to, where as usual $\Psi^{\pm}$ are uniquely defined by $\Psi^{\pm} = \exp\left(J_0 \int_0^t S(s)ds\right)$ (cf. (6.1)).

The Conley-Zehnder index will be defined as a map

$$\mathrm{CZ} : \mathcal{S}^*(2n) \to \mathbb{Z}.$$

Before defining it, let us first list its main properties. (the precise definition is for the moment less important). The Conley-Zehnder index was originally defined by Conley and Zehnder (surprise!) in [CZ84]. It was futher studied by Salamon-Zehnder [SZ92] and Robbin-Salamon[2] [RS93, RS95]. The following result comes from [SZ92].

Theorem 6.7 (Properties of the Conley-Zehnder index)**.** *The Conley-Zehnder index* $\mathrm{CZ} : \mathcal{S}^*(2n) \to \mathbb{Z}$ *satisfies:*

1. $\mathrm{CZ}(\Psi_0) = \mathrm{CZ}(\Psi_1)$ *if and only if* Ψ_0 *and* Ψ_1 *lie in the same connected component of* $\mathcal{S}^*(2n)$. *Thus* $\pi_0(\mathcal{S}^*(2n)) \cong \mathbb{Z}$, *with* $\mathrm{CZ} : \mathcal{S}^*(2n) \to \mathbb{Z}$ *furnishing an isomorphism.*
2. *If* $\Psi \in \mathcal{S}^*(2n)$ *and* $\Phi : S^1 \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$ *is a loop, then* $\Phi\Psi$ *also belongs to* $\mathcal{S}^*(2n)$ *and one has*

$$\mathrm{CZ}(\Phi\Psi) = \mathrm{CZ}(\Psi) + 2\mu(\Phi).$$

[2] Together with many other authors whose names I have omitted; this is purely due to laziness on my part.

3. *One has*
$$\operatorname{sgn}(\det(\mathbb{1} - \Psi(1)) = (-1)^{n-\mathrm{CZ}(\Psi)}.$$

4. *Suppose* $\Psi_1 \in \mathcal{S}^*(2n_1)$ *and* $\Psi_2 \in \mathcal{S}^*(2n_2)$. *Let* $\Psi = \Psi_1 \oplus \Psi_2 \in \mathcal{S}^*(2(n_1+n_2))$. *Then*
$$\mathrm{CZ}(\Psi) = \mathrm{CZ}(\Psi_1) + \mathrm{CZ}(\Psi_2).$$

5. *If* $\Theta : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$ *is any continuous path then*
$$\mathrm{CZ}(\Theta\Psi\Theta^{-1}) = \mathrm{CZ}(\Psi).$$

6. *If* $\Psi(t)$ *has no eigenvalues of modulus 1 for* $t \in (0,1]$ *then* $\mathrm{CZ}(\Psi) = 0$.

7. *For any* $\Psi \in \mathcal{S}^*(2n)$, *one has* $\mathrm{CZ}(\Psi^{-1}) = \mathrm{CZ}(\Psi^*) = -\mathrm{CZ}(\Psi)$.

8. *Suppose* $\Psi(t) = \exp(tJ_0S)$ *for some symmetric matric* S *satisfying* $\|S\| < 2\pi$*. Then*
$$\mathrm{CZ}(\Psi) = -\frac{1}{2}\operatorname{sgn} S.$$

In fact, properties (1), (2) and (3) above characterize the Conley-Zehnder index uniquely. There are several possible ways to define the Conley-Zehnder index. For our purposes, the most useful one is due to Robbin and Salamon [RS93]. This however was *not* the original definition. Suppose $\Psi(t) = \exp\left(J_0 \int_0^t S(s)ds\right)$ is a continuously differentiable path. Let us say a point $t \in [0,1)$ is a *crossing* if $\det(\mathbb{1} - \Psi(t)) = 0$. Suppose t is a crossing. Define the *crossing form*
$$\Gamma(\Psi, t) : \ker \mathbb{1} - \Psi(1) \to \mathbb{R}$$
by setting
$$\Gamma(\Psi,t)[v] := \omega_0(v, \Psi'(t)v).$$
Using $\Psi' = J_0S\Psi$, we can alternatively write
$$\Gamma(\Psi,t)[v] := -\langle S(t)v, v\rangle,$$
since $\Psi(t)v = v$ by assumption. We say that a crossing t is *regular* if the crossing form $\Gamma(\Psi,t)$ is a non-degenerate bilinear form. If t is a regular crossing then there exists $\varepsilon > 0$ such that there are no other crossings in $(t-\varepsilon, t+\varepsilon) \cap [0,1]$. The following result is due to Robbin and Salamon [RS93].

Theorem 6.8. *Suppose* $\Psi \in \mathcal{S}^*(2n)$ *is continuously differentiable and has only regular crossings. Then*
$$\mathrm{CZ}(\Psi) = \frac{1}{2}\operatorname{sgn}\Gamma(\Psi,0) + \sum_t \operatorname{sgn}\Gamma(\Psi,t),$$
where the sum is over all the crossings $t \in (0,1)$.

Unfortunately to apply Theorem 6.8 one has to know a priori that all the crossings are regular. Luckily, one has the following result, which is a special case of [RS93, Lemma 2.2].

Lemma 6.9. *Any path* $\Psi \in \mathcal{S}^*(2n)$ *can be uniformly approximated in* $\mathcal{S}^*(2n)$ *to a continuously differentiable one, all of whose crossings are regular.*

We will need the following two computations later.

Exercise 6.10. Take $n = 1$ and consider the path $\Psi_\theta(t) = \exp(tS_\theta J_0)$, where

$$S_\theta := -\theta \mathbb{1} = \begin{pmatrix} -\theta & 0 \\ 0 & -\theta \end{pmatrix}.$$

Show that Ψ_θ has only regular crossings, and check that

$$\mathrm{CZ}(\Psi_\theta) = 2\left\lfloor \frac{\theta}{2\pi} \right\rfloor + 1. \tag{6.4}$$

Similarly show that if $\Psi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then $\mathrm{CZ}(\Psi) = 0$.

Let us now for the sake of completeness give another definition of the Conley-Zehnder index. This definition is due to Salamon and Zehnder [SZ92], and has the advantage that one does not need to perturb Ψ in order to compute it. Unfortunately though it is not very practical for computations.

Theorem 6.11. *Fix any two matrics $W^\pm \in \mathrm{Sp}^\pm(\mathbb{R}^{2n}, \omega_0)$ that satisfy $\left(\widetilde{\det}(W^\pm)\right)^2 = 1$. Given any path $\Psi \in \mathcal{S}^*(2n)$, extend Ψ to a path $\tilde{\Psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$ such that $\tilde{\Psi}(t) \notin \mathrm{Sp}^0(\mathbb{R}^{2n}, \omega_0)$ for all $1 \le t \le 2$, and such that $\tilde{\Psi}(2) \in \{W^-, W^+\}$. Then one has*

$$\mathrm{CZ}(\Psi) = \mathrm{degree}\left(t \mapsto \left(\widetilde{\det}(\tilde{\Psi}(t))\right)^2\right). \tag{6.5}$$

Exercise 6.12. Show that the right-hand side of (6.5) is independent of the choice of $W^\pm$ and of the choice of extension $\tilde{\Psi}$. *Hint:* Use Exercise 6.4.

We now compute "by hand" the Fredholm index of a particular operator. Equip $\mathbb{C}\backslash\{0\}$ with cylindrical polar coordinates

$$\mathbb{C}\backslash\{0\} \cong \mathbb{R} \times S^1,$$
$$e^{2\pi(s+it)} \cong (s,t).$$

We can define the operator

$$\overline{\partial}_{J_0} : W^{1,p}(\mathbb{C}, \mathbb{R}^{2n}) \to L^p(\mathbb{C}, \mathbb{R}^{2n})$$

in exactly the same way as before:

$$\overline{\partial}_{J_0} = \partial_s + J_0 \partial_t.$$

Nevertheless in this section we wish to think of $\mathbb{C}$ as being obtained from the unit disc $\mathbb{D}$ by gluing on $[0, +\infty) \times S^1$:

$$\mathbb{C} = \mathbb{D} \cup_{\partial\mathbb{D}} ([0, +\infty) \times S^1).$$

Thus we continue to use the identification $(s,t) \cong e^{2\pi(s+it)}$ on $\mathbb{C}\backslash\mathbb{D}$ by on the interior of $\mathbb{D}$ we view $\mathbb{D}$ as a compact subset of $\mathbb{C}$, rather than as the non-compact Riemann surface $(-\infty, 0] \times S^1$. In other words, on $\mathbb{D}$ we wish to use the measure $r dr \wedge d\varphi$ in polar coordinates

$(r,\varphi) \in (0,1)\times\mathbb{R}\backslash 2\pi\mathbb{Z}$, rather than the measure $ds\wedge dt$. One way to do this is to introduce a smooth cut-off function

$$\alpha : \mathbb{R} \to [0,+\infty),$$
$$\alpha(s) = \begin{cases} \frac{1}{2\pi e^{2\pi s}}, & s \le -1, \\ 1, & s \ge 1, \end{cases}$$
$$\alpha'(s) \le 0.$$

Now equip $\mathbb{C}\backslash\{0\}$ with the measure

$$d\mu = \frac{1}{\alpha(s)^2} \cdot ds \wedge dt.$$

Thus on the cylindrical end we obtain the standard measure $ds\wedge dt$, whereas on the interior of the disc $\{e^{2\pi(s+it)} \mid s \le -1\}$ we obtain the measure $rdr \wedge d\varphi$. Now define

$$D_0^+ : C^\infty(\mathbb{C}, \mathbb{R}^{2n}) \to C^\infty(\mathbb{C}, \mathbb{R}^{2n})$$

by

$$D_0^+ = \alpha(s)\overline{\partial}_{J_0}.$$

Suppose now we are given a matrix valued function $S : [0,+\infty) \times S^1 \to \mathsf{L}(\mathbb{R}^{2n})$ with the property that$\lim_{s\to+\infty} S(s,t) =: S^+(t)$ is a symmetric loop of matrices whose associated fundamental solution $\Psi^+ : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$ belongs to $\mathcal{S}^*(2n)$, and such that $S(s,t) = 0$ for all $(s,t) \in [0,1] \times S^1$. Then it makes sense to study the operator

$$D_S^+ := D_0^+ + \alpha(s)S.$$

The notation '+' is meant to indicate that we are viewing $\mathbb{C}$ here as obtained by obtaining a *positive* cylindrical end to $\mathbb{D}$. Now let

$$L^p_{d\mu}(\mathbb{C}, \mathbb{R}^{2n}) := \left\{ u \in L^p_{\mathrm{loc}}(\mathbb{C}, \mathbb{R}^{2n}) \mid \int_{\mathbb{C}} |u|^p \, d\mu < \infty \right\},$$

and define

$$W^{1,p}_{d\mu}(\mathbb{C}, \mathbb{R}^{2n}) := \left\{ u \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, \mathbb{R}^{2n}) \mid u, u' \in L^p_{d\mu}(\mathbb{C}, \mathbb{R}^{2n}) \right\},$$

We now prove:

Theorem 6.13. *For any* $1 < p < +\infty$ *the operator*

$$D_S^+ : W^{1,p}_{d\mu}(\mathbb{C}, \mathbb{R}^{2n}) \to L^p_{d\mu}(\mathbb{C}, \mathbb{R}^{2n})$$

is a Fredholm operator of index

$$\mathrm{ind}\, D_S^+ = n - \mathrm{CZ}(\Psi^+).$$

Proof. The proof that D_S^+ is Fredholm can be carried out in a similar fashion to the proof of Theorem 5.6, and thus may be safely left as an exerise to the reader:

Exercise 6.14. Prove that D_S^+ is indeed Fredholm.

In order to calculate the index we proceed in two steps.

Step 1: We prove the result in the case $n = 1$ in two special cases: take $S(s,t) = \beta(s)S^+(t)$, where

$$\beta : [0,+\infty) \to [0,1]$$

satsifies $\beta(1) = 0$ and $\beta(2) = 1$, with $\beta' \ge 0$. Moroever we take $S^+(t)$ to be the constant matrix:

1. $S_\theta = \begin{pmatrix} -\theta & 0 \\ 0 & -\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$,

2. $S^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

By Exercise 6.10, in Case (1) we have $\mathrm{CZ}(\Psi^+) = 2\left\lfloor \frac{\theta}{2\pi} \right\rfloor + 1$, and in Case (2) we have $\mathrm{CZ}(\Psi^+) = 0$, and thus we must show that in Case (1) one has $\mathrm{ind}\, D_S^+ = -2\left\lfloor \frac{\theta}{2\pi} \right\rfloor$ and in Case (2) we have $\mathrm{ind}\, D_S^+ = 1$. In fact, we will prove only the following statement:

Lemma 6.15. *Given $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$, if $D_\theta^+ := D_S^+$ for $S = S_\theta$ as in Case (1) above, one has*

$$\dim\ker D_\theta^+ = \begin{cases} 0, & \theta > 0, \\ 2\left\{k \in \mathbb{Z} \mid 0 \leq k < -\theta/2\pi\right\}, & \theta < 0. \end{cases}$$

Proof. If $u \in C^\infty_{d\mu}(\mathbb{C}, \mathbb{R}^{2n})$ belongs to the kernel of D_S^+ then we can expand u in a Fourier series:

$$u(s,t) = \sum_{k \in \mathbb{Z}} u_k(s) e^{2\pi k i t}, \quad u_k : \mathbb{R} \to \mathbb{C}.$$

Then one has

$$0 = D_S^+ u = \begin{cases} \frac{1}{2\pi e^{2\pi s}}(\partial_s + i\partial_t)\sum_k u_k(s) e^{2\pi k i t}, & s \leq -1, \\ (\partial_s + i\partial_t - \theta\mathbb{1})\sum_k u_k(s) e^{2\pi k i t}, & s \geq 1. \end{cases}$$

which tells us that the functions u_k must satisfy the relations

$$\sum_{k \in \mathbb{Z}} \frac{1}{2\pi e^{2\pi s}}\left(u_k'(s) - 2\pi k u_k(s)\right) e^{2\pi k i t} = 0, \quad s \leq -1,$$

$$\sum_{k \in \mathbb{Z}} \left(u_k'(s) - 2\pi k u_k(s) - \theta u_k(s)\right) e^{2\pi k i t} = 0, \quad s \geq 1.$$

This tells us that:

$$u_k'(s) - 2\pi k u_k(s) = 0, \quad , s \leq -1,$$
$$u_k'(s) - (2\pi k + \theta) u_k(s) = 0, \quad s \geq 1.$$

In other words, we must have

$$u_k(s) = a_k e^{2\pi k s}, \quad s \leq -1, \tag{6.6}$$

$$u_k(s) = a_k e^{(2\pi k + \theta)s}, \quad s \geq 1. \tag{6.7}$$

for some constants $a_k \in \mathbb{R}$. Let us take $p = 2$ (cf. Lemma 5.13. The requirement that $u \in W^{1,2}_{d\mu}(\mathbb{C}, \mathbb{C})$ implies that

$$\int_{-\infty}^{-1} \left(\frac{1}{4\pi^2 e^{4\pi s}}(1 + 4\pi^2 k^2)\left|u_k\right|^2 + \left|u_k'\right|^2 \right) ds < \infty, \tag{6.8}$$

$$\int_1^{+\infty} \left((1 + 2\pi k)\left|u_k\right|^2 + \left|u_k'\right|^2 \right) ds < \infty. \tag{6.9}$$

Plugging in (6.6) to (6.8) we see that

$$\int_{-\infty}^{-1} \left(\frac{1}{4\pi^2 e^{4\pi s}}(1 + 4\pi^2 k^2) a_k^2 e^{4\pi k s} + a_k^2 4\pi^2 k e^{4\pi k s} \right) ds < +\infty,$$

which implies that we must have $a_k = 0$ for $k \leq 0$ $k \geq 0$.

Similarly plugging (6.7) into (6.9) we see that

$$\int_1^{+\infty} \left((1+2\pi k) a_k^2 e^{(4\pi k + 2\theta)s} + a_k^2 (2\pi k + \theta)^2 e^{(4\pi k + 2\theta)s} \right) ds < +\infty,$$

which implies we must have$a_k = 0$ unless $4\pi k + 2\theta < 0$. It follows that

$$\dim_{\mathbb{C}} \ker D_S^+ = \# \{ k \in \mathbb{Z} \mid 0 \leq k < -\theta/2\pi \}.$$

Thus the real dimension is twice this, and the lemma follows. ■

Exercise 6.16. Complete the proof of Step 1. *Hint:* Use the fact that the adjoint of S_θ is $S_{-\theta}$ to deal with Case (1). Case (2) is easier.

Step 2: We now complete the proof. Suppose to begin with that $n \geq 2$. Then if $\mathrm{CZ}(\Psi^+) = k$ we can choose $\theta_1, \dots, \theta_m \in \mathbb{R} \backslash 2\pi\mathbb{Z}$ for some $1 \leq m \leq n$ such that

$$k = \left(2 \left\lfloor \frac{\theta_1}{2\pi} \right\rfloor + 1 \right) + \cdots + \left(2 \left\lfloor \frac{\theta_m}{2\pi} \right\rfloor + 1 \right).$$

Let $\Psi_i \in \mathcal{S}^*(2n)$ denote the path $\Psi_i(t) = \exp(-t\theta_i J_0)$ for $i = 1, \dots, m$ and let $\Psi_i(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ for $i = m+1, \dots, n$. Let $\tilde{\Psi} = \Psi_1 \oplus \cdots \oplus \Psi_n$. Then from Property (4) of Theorem 6.7 and Exercise 6.10 one has $\mathrm{CZ}(\tilde{\Psi}) = k$. If we write

$$\tilde{\Psi}(t) = \exp \left(J_0 \int_0^t \tilde{S}(s) ds \right),$$

then it follows from an argument analogous to Exercise 6.6 that

$$\operatorname{ind} D_S^+ = \operatorname{ind} D^+_{\beta(s)\tilde{S}(t)}.$$

Moreover the argument of Step 1 shows that

$$\operatorname{ind} D^+_{\beta(s)\tilde{S}(t)} = n - \mathrm{CZ}(\tilde{\Psi}).$$

Thus

$$\operatorname{ind} D_S^+ = \operatorname{ind} D^+_{\beta(s)\tilde{S}(t)} = n - \mathrm{CZ}(\tilde{\Psi}) = n - \mathrm{CZ}(\Psi^+),$$

which completes the proof of Theorem 6.7. ■

We now recall the statement of the *Riemann-Roch Theorem*. Suppose Σ is a closed Riemann surface, and let j denote an (integrable) almost complex structure on Σ. Suppose $(E, J) \to (\Sigma, j)$ is a smooth complex vector bundle of complex rank m.

Definition 6.17. A *real linear Cauchy-Riemann operator* D is an operator

$$D : \Gamma(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)$$

which is $\mathbb{R}$-linear and which satisfies the Leibniz rule

$$D(f \cdot \xi) = f \cdot D\xi + \overline{\partial} f \cdot \xi \tag{6.10}$$

for $f : \Sigma \to \mathbb{R}$ and $\xi \in \Omega^0(\Sigma, E)$. Here $\overline{\partial} : C^\infty(\Sigma) \to \Omega^{0,1}(\Sigma)$ is the composition of the $\overline{\partial} f = \pi^{0,1}(df)$.

If one asks that D is $\mathbb{C}$-linear and that (6.10) also holds for *complex* valued f (in which case we say D is a *complex linear Cauchy Riemann* operator) then a deep theorem says that complex linear Cauchy-Riemann operators are in 1-1 correspondence with holomorphic structures on E.

Theorem 6.18 (The Riemann-Roch Theorem)**.** *A Cauchy-Riemann operator extends to a well defined operator*

$$D : W^{1,p}(\Sigma, E) \to L^p(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$$

which is Fredholm of index

$$\operatorname{ind} D = m\chi(\Sigma) + 2\langle c_1(E), [\Sigma]\rangle .$$

Exercise 6.19. Look up the classical statement of the Riemann-Roch Theorem in an algebraic geometry textbook. Reconcile the statement there with the one above. *Hint:* Every complex vector bundle is isomorphic to a holomorphic one. If D is a complex linear Cauchy Riemann operator then the index of D can be identified with the dimension of certain Dolbeault cohomology groups. Look at [MS12, Remark C.1.11] for more information.

Later on we will need more to use the full statement of Theorem 6.18, but for now we need only the following corollary.

Corollary 6.20. *Consider the operator*

$$\overline{\partial} : C^\infty(S^2, \mathbb{R}^{2n}) \to \Omega^{0,1}(S^2, \mathbb{R}^{2n}),$$

$$\overline{\partial} u := du + j \circ du \circ j.$$

Then $\overline{\partial}$ is Fredholm of index $2n$.

We can now finally prove our main result:

Theorem 6.21. *Fix $1 < p < +\infty$ and suppose $S : \mathbb{R} \times S^1 \to \mathsf{L}(\mathbb{R}^{2n})$ is a smooth map such that the limits $S^\pm(t) := \lim_{s\to\pm t} S(s,t)$ exist and the convergence is uniform in t. Suppose that $S^\pm$ are symmetric, and that the assocaited symplectic matrices $\Psi^\pm(t) = \exp\left(J_0 \int_0^t S(s)ds\right)$ belong to $\mathcal{S}^*(2n)$. Then the operator*

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is Fredholm of index

$$\operatorname{ind} D_S = \mathrm{CZ}(\Psi^-) - \mathrm{CZ}(\Psi^+).$$

Proof. Consider an operator $D^+_{S^-} = D^-_{\beta(s)S^-(t)} : W^{1,p}_{d\mu}(\mathbb{C}, \mathbb{R}^{2n}) \to L^p_{d\mu}(\mathbb{C}, \mathbb{R}^{2n})$ as in Theorem 6.13. Then $\operatorname{ind} D^+_{S^-} = n - \mathrm{CZ}(\Psi^-)$. Similarly we can form another operator $D^-_{S^+}$ defined on $\mathbb{C}$, this time thought of attaching a *negative* cylindrical end to $\mathbb{D}$. Arguments entirely similar to those of Theorem 6.13 show that $D^-_{S^+}$ is Fredholm of index $n + \mathrm{CZ}(\Psi^+)$. Next, if we glue all three operators together we obtain an operator

$$D = D^+_{S^-} \# D_S \# D^-_{S^+} : W^{1,p}(S^2, \mathbb{R}^{2n}) \to L^p(S^2, \mathbb{R}^{2n}),$$

whose index satisfies

$$\operatorname{ind} D = \operatorname{ind} D^+_{S^-} + \operatorname{ind} D_S + \operatorname{ind} D^-_{S^+},$$

by Theorem 5.15. This operator agrees with the operator $\overline{\partial}$ from Corollary 6.20 away from a compact subset of S^2, and hence the Fredholm index of D is the same as that of $\overline{\partial}$. Thus

$$\operatorname{ind} D_S = 2n - (n - \mathrm{CZ}(\Psi^-)) - (n + \mathrm{CZ}(\Psi^+)) = \mathrm{CZ}(\Psi^-) - \mathrm{CZ}(\Psi^+)$$

as required. ■

We now define the index $\mathrm{CZ}(x)$ of a solution $x \in \mathcal{P}_1(H)$.

Definition 6.22. Fix $x \in \mathcal{P}_1(H)$ and choose an admissible symplectic trivialisation $\Phi : S^1 \times \mathbb{R}^{2n} \to x^*TQ$ (in the sense of Definition 3.18). Define the symplectic path $\Psi \in \mathcal{S}^*(2n)$ by

$$\Psi(t) := \Phi_t^{-1} \circ D\phi_H^t \circ \Phi_0. \tag{6.11}$$

Then define the *Conley-Zehnder index* of x, written $\mathrm{CZ}(x)$, by

$$\mathrm{CZ}(x) := \mathrm{CZ}(\Psi).$$

Exercise 6.23. Show that the definition of $\mathrm{CZ}(x)$ is well defined (i.e. independent of the choice of admissible symplectic trivialisation). *Hint:* Use Lemma 3.19.

We can now summarise our work in Section 5 and Section 6.

Theorem 6.24. *Let $H \in \mathcal{H}_{\mathrm{reg}}$ and choose $J \in \mathcal{J}(Q, \omega)$. Fix a pair x^-, x^+ of elements of $\mathcal{P}_1(H)$. Then if $u \in \mathcal{M}(x^-, x^+)$ is a zero of the Floer operator $\partial_{J,H}$ from Definition 4.3 then the vertical derivative $D^v\partial_{J,H}(u)$ is a Fredholm operator of index $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$.*

CHAPTER 7

Transversality for the moduli spaces

In this section we show that, after a further perturbation of the Hamiltonian H away from the set of points in M lying on the image of an element of $\mathcal{P}_1(H)$, the section $\partial_{J,H}$ is onto at every solution $u \in \mathcal{M}(x^-, x^+)$, for every pair $x^\pm$ of elements of $\mathcal{P}_1(H)$.

Definition 7.1. Let $H_0 \in \mathcal{H}_{\text{reg}}$ denote a Hamiltonian with the property that all of the elements of $\mathcal{P}_1(H_0)$ are non-degenerate. Denote by $\mathcal{H}(H_0)$ denote the subset of $C^\infty(S^1 \times Q)$ of functions H such that H agrees with H_0 up to second order on the set $\{x(S^1) \mid x \in \mathcal{P}_1(H)\} \subset Q$.

Here is the main result of this section.

Theorem 7.2. *Let $H_0 \in \mathcal{H}_{\text{reg}}$ and fix $J \in \mathcal{J}(Q, \omega)$. Let $\mathcal{H}^\partial(H_0, J)$ denote the subset of functions $H \in \mathcal{H}(H_0)$ with the property that for every pair x^-, x^+ of elements in $\mathcal{P}_1(H_0)$, and every solution $u \in \mathcal{M}(x^-, x^+)$, the operatore $D^v\partial_{J,H}(u)$ is surjective and admits a right inverse. Then the set $\mathcal{H}^\partial(H_0, J)$ is of second category in $\mathcal{H}(H_0)$.*

Note that by the Implicit Function Theorem 1.6 and Theorem 6.24, if $H \in \mathcal{H}^\partial(H_0, J)$ then for any two orbits x^- and x^+ in $\mathcal{P}_1(H_0)$, the moduli spaces $\mathcal{M}(x^-, x^+)$ are all manifolds of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$.

Remark 7.3. Note that if $H \in \mathcal{H}(H_0)$ is such that $\|H - H_0\|_{C^2(S^1 \times Q)}$ is sufficiently small then $\mathcal{P}_1(H) = \mathcal{P}_1(H_0)$ and thus in particular $H \in \mathcal{H}_{\text{reg}}$. For the purposes of defining the Floer groups associated to the Hamiltonian system determined by $H_0 \in \mathcal{H}_{\text{reg}}$ (i.e. the Floer groups whose generators are the elements of $\mathcal{P}_1^\circ(H_0)$), it is sufficient to work with any element H in the set

$$\mathcal{H}^\partial_{\text{reg}}(H_0) := \Big\{ H \in \mathcal{H}^\partial(H_0, J) \mid \mathcal{P}_1(H) = \mathcal{P}_1(H_0) \Big\}$$

The proof of Theorem 7.2 is similar to the proof of Theorem 3.21, and involves an application of the Transversality Theorem 3.22. However this result is considerably more difficult: this should be compared to case of Morse theory, where it is much harder to show that a generic pair (f, g) is Morse-Smale (Theorem 1.18) than it is to show that a generic function f is Morse. The proof we will give is due to Salmon and Zehnder [SZ92]. However the main step in the proof was not rigourously proved until a few years later, by Floer, Hofer, and Salamon [FHS96]. This step is the following result on the existence of *regular points* of solutions of the Floer equation.

Definition 7.4. Fix a pair $x^\pm$ of elements of $\mathcal{P}_1(H)$ and an element $u \in \mathcal{M}(x^-, x^+)$. A point $(s, t) \in \mathbb{R} \times S^1$ is called a *regular point* of u if:

$$\partial_s u(s,t) \neq 0, \qquad u(s,t) \neq x^\pm(t), \qquad u(s,t) \neq u(s',t) \text{ for some } s' \neq s.$$

We denote by $R(u)$ the set of regukar points of u. Note that if (s, t) is a regular point of u then the curve $\sigma \mapsto u(\sigma, t)$ is an immersion for σ near s and meets the point $u(s, t)$ only once.

The reason why we require more than just $\partial_s u(s, t) \neq 0$ in the definition of regular points is provided by the following exercise.

Exercise 7.5. Suppose $(s_0, t_0) \in R(u)$. Prove that there exists a neighborhood $U_0 \subset S^1 \times M$ of $(t_0, u(s_0, t_0))$ such that

$$V_0 := \left\{(s,t) \in \mathbb{R} \times S^1 \mid (t, u(s,t)) \in U_0\right\}$$

is a neighborhood of (s_0, t_0) in $\mathbb{R} \times S^1$. Similarly prove that for any smooth function $k : \mathbb{R} \times S^1 \to \mathbb{R}$ which is supported in a sufficiently small neighborhood of (s_0, t_0), there exists a smooth cutoff function $\nu : S^1 \times Q \to \mathbb{R}$ such that

$$\nu(t, u(s,t)) = k(s,t), \qquad \text{for all } (s,t) \in \mathbb{R} \times S^1. \tag{7.1}$$

Here is the key result of Floer, Hofer and Salamon [FHS96] mentioned above that we will use to prove Theorem 7.2.

Theorem 7.6. *If $x^- \neq x^+$ then the set $R(u)$ is always an open, dense subset of $\mathbb{R} \times S^1$.*

For now let us assume Theorem 7.6 and show it can be used with the Tranversality Theorem 3.22 to prove Theorem 7.2.

Proof of Theorem 7.2. Fix $p > 2$ and $k \geq 2$, and consider the section

$$\sigma : \mathcal{B}^{1,p}(x^-, x^+) \times \mathcal{H}^k(H_0) \to \mathcal{E}^p$$

defined by

$$\sigma(u, H) := \partial_{J,H}(u),$$

where $\mathcal{B}^{1,p}(x^-, x^+)$ was defined in Definition 4.4 and $\mathcal{E}^p$ was defined in equation (4.3), and $\mathcal{H}^k(H_0)$ is defined in the same way as $\mathcal{H}(H_0)$, only working with functions of class C^k instead of C^∞. Note that the tangent space to $\mathcal{H}^k(H_0)$ is simply the set of function $\hat{H} \in C^k(S^1 \times Q)$ that vanish to second order over points in Q lying over elements of $\mathcal{P}_1(H_0)$. The vertical derivative of this section at a zero (u, H) is the operator

$$D^v\sigma(u, H) := W^{1,p}(u^*TQ) \times T_H\mathcal{H}(H_0) \to L^p(u^*TQ),$$

$$D^v\sigma(u, H)[\hat{u}, \hat{H}] = D^v\partial_{J,H}(u)[\hat{u}] - \nabla\hat{H}_t(u).$$

Here

$$D^v\partial_{J,H}(u)[\hat{u}] = \nabla_s\hat{u} + J(u)\nabla_t\hat{u} + \nabla_{\hat{u}}J(u)\partial_t u - \nabla_{\hat{u}}\nabla H_t(u). \tag{7.2}$$

Here we warn the reader not to be misled: the symbol "∇" refers to both the Levi-Civita connection of the metric $\omega(J\cdot, \cdot)$, as well as the gradient with respect to this metric (so that $\nabla H_t(u) = J(u)X_{H_t}(u)$.)

Exercise 7.7. Verify (7.2) by arguing as in the proof of Lemma 3.15.

By the Transversality Theorem 3.22, it is sufficient to show that $D^v\sigma(u, H)$ is surjective, or equivalently, that the annhilator of its image vanishes. In other words, we must show that if $\hat{w} \in L^q(u^*TQ)$, where $1/p + 1/q = 1$, satisfies

$$\int_{\mathbb{R}}\int_{S^1} \left\langle D^v\sigma(u, H)[\hat{u}, \hat{H}], \hat{w}\right\rangle_J dtds = 0, \qquad \text{for all } (\hat{u}, \hat{H}) \in W^{1,p}(\hat{u}^*TQ) \times T_H\mathcal{H}^k(H_0),$$

then $\hat{w} \equiv 0$. Note that any such $\hat{w}$ is in particular a weak solution of the equation

$$(D^v\overline{\partial}_{J,H}(u))^*[\hat{w}] = 0,$$

and hence by elliptic regularity (Theorem 4.19), any such $\hat{w}$ is neccesarily a strong solution, and is smooth.

Remark 7.8. In the proof of Theorem 7.6 below we shall show that any such solution $\hat{w}$ satisfies the "principle of analytic continuation" property enjoyed by holomorphic maps: namely that if $\hat{w}$ vanishes on an open subset of $\mathbb{R} \times S^1$ then $\hat{w}$ vanishes everywhere (see Proposition 7.14). In fact, we will show that $\hat{w}$ vanishes on the open subset $R(u)$; since this set is also dense by Theorem 7.6 we do not actually need to use the principle of analytic continuation here in order to conclude. Nevertheless it is perhaps helpful to remark that in theory we need only show that $\hat{w}$ vanishes on an open set.

We will now prove that any smooth map $\hat{w} \in L^q(u^*TQ)$ that satisfies

$$\int_{\mathbb{R}} \int_{S^1} d\hat{H}_t(u)[\hat{w}]dtds = 0, \qquad \text{for all } \hat{H} \times T_H \mathcal{H}^k(H_0), \tag{7.3}$$

is in fact, identically zero. The strategy of the proof is thus:

1. We first show that if $C(u) := \{(s,t) \in \mathbb{R} \times S^1 \mid \partial_s u(s,t) = 0\}$ then there exists a unique C^k-function $\alpha : (\mathbb{R} \times S^1) \backslash C(u) \to \mathbb{R}$ such that

$$\hat{w}(s,t) = \alpha(s,t)\partial_s u(s,t).$$

2. We then show that $\partial_s \alpha(s,t) = 0$, and hence as $C(u)$ is discrete (cf. Lemma **??**), α extends to a unique C^k-function $\overline{\alpha} : S^1 \to \mathbb{R}\backslash\{0\}$ such that

$$\hat{w}(s,t) = \overline{\alpha}(t)\partial_s u(s,t). \tag{7.4}$$

3. We show that (7.4) contradicts the assertion that $\hat{w} \in L^q(u^*TQ)$, thus completing the proof.

In order to prove (1) it suffices to obtain a contradiction from the hypothesis that there exists a point $(s_0,t_0) \in R(u)$ such that $\hat{w}(s_0,t_0)$ and $\partial_s u(s_0,t_0)$ are linearly independent. Since $(s_0,t_0) \in R(u)$, by Exercise 7.5, there exists a neighborhood $U_0 \subset S^1 \times M$ of $(t_0, u(s_0,t_0))$ such that

$$V_0 := \{(s,t) \in \mathbb{R} \times S^1 \mid (t, u(s,t)) \in U_0\}$$

is a neighborhood of (s_0,t_0) in $\mathbb{R} \times S^1$. Consider now function $f_t : B((0,s_0);\varepsilon) \to U_0$ (here $B((0,s_0);\varepsilon)$ denotes the disc of radius $\varepsilon > 0$ in $\mathbb{R}^2$ about the point $(0,s_0)$) defined by

$$f_t(r,s) := \exp_{u(s,t)}(r\hat{w}(s,t)).$$

By choice of U_0 the function f_t is an embedding satsifying

$$f_t(0,s) = \partial_s u(s,t), \qquad \partial_r f_t(0,s) = \hat{w}(s,t).$$

Conisder now a cutoff function $\beta : \mathbb{R} \to [0,1]$ which is equal to 1 on a small neighborhood of 0 and zero elsewhere. Since f_t is an embedding and $(s_0,t_0) \in R(u)$, we can find $\hat{H} \in T_H\mathcal{H}^k(H_0)$ such that $\hat{H}$ is supported in U_0 and such that $\hat{H}_t$ satisfies

$$\hat{H}_t(f_t(r,s)) = r\beta(r)\beta(s-s_0)\beta(t-t_0).$$

Differentiating with respect to r at $r=0$ we see that

$$d\hat{H}_t(u(s,t))[\hat{w}(s,t)] = \beta(s-s_0)\beta(t-t_0),$$

and hence

$$\int_{\mathbb{R}}\int_{S^1} d\hat{H}_t(u(s,t))[\hat{w}(s,t)]dsdt = \int\int_{U_0} d\hat{H}_t(u(s,t))[\hat{w}(s,t)]dsdt > 0,$$

thus contradicting (7.3). This finishes the proof of (1). The proof of (2) is similar: if $\partial_s\alpha \neq 0$ then there exists a point $(s_0,t_0) \in R(u)$ and a smooth function $k : \mathbb{R}\times S^1 \to \mathbb{R}$ with support in a neighborhood V_0 of (s_0,t_0) as above such that

$$\int\int_{V_0} \alpha\partial_s k dsdt = \int\int_{V_0} \partial_s\alpha k dsdt \neq 0.$$

Now by Exercise 7.5 we can choose a function $\nu : S^1 \times Q \to \mathbb{R}$ such that equation (7.1) holds:

$$\nu(t,u(s,t)) = k(s,t).$$

Set $\hat{H}_t := \nu(t,\cdot)$. Then we have

$$\int_{\mathbb{R}}\int_{S^1} d\hat{H}_t(u(s,t))[\hat{w}(s,t)]dsdt > 0,$$

which again contradicts (7.3). Finally to prove (3) we may assume without loss of generality that $\overline{\alpha}(t) > 0$ for all t. We claim that

$$\frac{d}{ds}\int_0^1 \langle \partial_s u(s,t), \hat{w}(s,t)\rangle_J\, dt = 0. \tag{7.5}$$

Indeed, since

$$D^v\overline{\partial}_{J,H}(u)[\partial_s u] = 0, \qquad (D^v\overline{\partial}_{J,H}(u))^*[\hat{w}] = 0,$$

we have that in a suitable symplectic trivialisation of u^*TQ, if $\hat{v}_1(s,t)$ corresponds to $\partial_s u(s,t)$ and $\hat{v}_2(s,t)$ corresponds to $\hat{w}(s,t)$, then

$$\partial_s\hat{v}_1 + J_0\partial_t\hat{v}_2 + S\hat{v}_1 = 0, \qquad \partial_s\hat{v}_2 - J_0\partial_t\hat{v}_2 - S^*\hat{v}_2 = 0$$

for some matrix valued function S (see equation (5.1)). Then

$$\begin{aligned}\frac{d}{ds}\int_0^1 \langle \hat{v}_1,\hat{v}_2\rangle\, dt &= \int_0^1 \langle \partial_s\hat{v}_1,\hat{v}_2\rangle + \langle \hat{v}_1,\partial_s\hat{v}_2\rangle\, dt\\ &= \int_0^1 \langle -J_0\partial_t\hat{v}_2 - S\hat{v}_1,\hat{v}_2\rangle + \langle \hat{v}_1, J_0\partial_t\hat{v}_2 + S^*\hat{v}_2\rangle\, dt\\ &= -\int_0^1 \partial_t\langle J_0\hat{v}_1,\hat{v}_2\rangle\, dt = 0.\end{aligned}$$

But (7.5) implies that

$$\int_{\mathbb{R}}\int_{S^1} \langle \partial_s u,\hat{w}\rangle_J\, dsdt = +\infty,$$

since

$$\int_{S^1} \langle \partial_s u,\hat{w}\rangle_J\, dt = \int_{S^1} \overline{\alpha}(t)\,|\partial_s u(s,t)|_J^2\, dt > 0.$$

This contradicts the fact that $\partial_s u \in L^p(u^*TQ)$ and $\hat{w} \in L^q(u^*TQ)$, and thus finally completes the proof of Theorem 7.2. ■

We will now work towards proving the so-called *principle of analytic continuation*. The material in this section is taken from [FHS96]. In what follows we denote by $B_\varepsilon = B(0;\varepsilon) \subset \mathbb{C}$ the disc of radius ε. We write $z = s + it$. We denote by $\mathsf{L}_\mathbb{R}(\mathbb{C}^n)$ the set of $\mathbb{R}$-linear maps $\mathbb{C}^n \to \mathbb{C}^n$.

Proposition 7.9 (The Carleman Similarity Principle). *Let $J \in W^{1,p}(B_\varepsilon, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$ with $J(z)^2 = -\mathbb{1}$ for each $z \in B_\varepsilon$, and let $S \in L^p(B_\varepsilon, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$, and suppose $u \in W^{1,p}(B_\varepsilon, \mathbb{C}^n)$ solves the equation*

$$\partial_s u(z) + J(z)\partial_t u(z) + S(z)u(z) = 0, \qquad u(0) = 0.$$

Then there exists $0 < \delta < \varepsilon$ and a map $\Phi \in W^{1,p}(B_\delta, \mathsf{GL}_\mathbb{R}(\mathbb{C}^n))$ such that

$$J(z)\Phi(z) = \Phi(z)i,$$

together with a holomorphic map $w : B_\delta \to \mathbb{C}^n$ such that $w(0) = 0$ and such that

$$u(z) = \Phi(z)w(z), \qquad \text{for all } z \in B_\delta.$$

Proof. First choose $0 < \delta_1 < \varepsilon$ and a map $\Psi \in W^{1,p}(B_{\delta_1}, \mathsf{GL}_\mathbb{R}(\mathbb{C}^n))$ such that

$$\Psi(z)J(z) = \Psi(z)i, \qquad \text{for all } z \in B_{\delta_1}.$$

Exercise 7.10. Prove such a Ψ exists.

We now define $v \in W^{1,p}(B_{\delta_1}, \mathbb{C}^n)$ by requiring that $u(z) = \Psi(z)v(z)$. We would be done if v was holomorphic, but sadly this is unlikely to be the case. Indeed, we compute that

$$0 = \partial_s u + J\partial_t u + Su = \Psi(\partial_s v + i\partial_t v + Tv), \tag{7.6}$$

where

$$T := \Psi^{-1}(\partial_s\Psi + J\partial_t\Psi + S\Psi) \in L^p(B_{\delta_1}, \mathsf{L}_\mathbb{R}(\mathbb{C}^n)).$$

Our aim is thus to modify Ψ so that the T term disappears. Let us split T into its complex linear and anti-linear parts:

$$T^\pm(z) := \frac{1}{2}(T(z) \mp iT(z)i).$$

We multiply T^- by a new map B such that if $A := T^+ + T^-B$ then A is complex linear and satsifies $T(z)v(z) = A(z)v(z)$ for all $z \in B_\delta$. For instance, we could take

$$B(z)[\zeta] := \begin{cases} \frac{1}{|v(z)|^2} v(z) \cdot v(z)^\mathsf{T} \cdot \bar{\zeta}, & v(z) \neq 0, \\ 0, & v(z) = 0. \end{cases}$$

Given $0 < \delta < \delta_1$, let A_δ denote the operator that coincides with A on B_δ and vanishes outside of B_δ. We regard A_δ as a map defined on S^2; note that $A_\delta \in L^p(S^2, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$.Now consider the Cauchy-Riemann operator

$$D_\delta : W^{1,p}(S^2, \mathsf{L}_\mathbb{R}(\mathbb{C})) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes \mathsf{L}_\mathbb{R}(\mathbb{C}^n)).$$

defined by

$$D_\delta U = \overline{\partial}U + A_\delta U d\overline{z}, \quad U \in W^{1,2}(S^2, \mathsf{L}_\mathbb{R}(\mathbb{C}^n)).$$

By Corollary 6.20 , we see that D_δ is a Fredholm operator of index $2\dim_\mathbb{C} \mathsf{L}_\mathbb{R}(\mathbb{C}^n)$.

Exercise 7.11. Prove that the map

$$\Theta_\delta : W^{1,p}(S^2, \mathsf{L}_\mathbb{R}(\mathbb{C}^n)) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes \mathsf{L}_\mathbb{R}(\mathbb{C}^n)) \times \mathsf{L}_\mathbb{R}(\mathbb{C}^n)$$

defined by

$$\Theta_\delta(U) := (\bar\partial U, U(0))$$

is a bijective. *Hint:* Use Corollary 6.20, together with the analgoue of Liouville's Theorem from complex analysis.

Since the operator A_δ satisfies $\lim_{\delta\to 0} \|A_\delta\|_{L^p(S^2, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))} = 0$, it follows that for δ sufficiently the map $U \mapsto (D_\delta U, U(0))$ is also bijective. Thus for $\delta > 0$ sufficiently small there exists a function $E_\delta \in W^{1,p}(S^2, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$ such that

$$D_\delta E_\delta = 0, \qquad E_\delta(0) = \mathbb{1}.$$

Moreover since E_δ converges to the constant map as $\delta \to 0$ in the $W^{1,p}$-norm, we see that for $\delta > 0$ sufficiently small that complex linear map E_δ is invertible. Fix such a δ and set $\Phi(z) := \Psi(z)E_\delta(z)$ and $w(z) := E_\delta(z)^{-1}v(z)$. Then clearly $\Phi \in W^{1,p}(B_\delta, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$ and $J(z)\Phi(z) = \Phi(z)i$ for all $z \in B_\delta$. Moreover we certainly have $u(z) = \Phi(z)w(z)$ in B_δ. It remains to show that w is holomorphic. Since

$$\partial_s E_\delta + i\partial_t E_\delta + A_\delta E_\delta = 0,$$

we see from (7.6) that

$$0 = \partial_s v + i\partial_t v + Tv = E_\delta(\partial_s w + i\partial_t w).$$

Thus w is holomorphic. ∎

Corollary 7.12. *Let $k \geq 2$ and $p > 2$ and suppose $J \in W^{k,p}(B_\varepsilon, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$ satisfies $J(z)^2 = -\mathbb{1}$ for each $z \in B_\varepsilon$. Let $S \in W^{k-1,p}(B_\varepsilon, \mathsf{L}_\mathbb{R}(\mathbb{C}^n))$ and suppose $u \in W^{k,p}(B_\varepsilon, \mathbb{C}^n)$ solves the equation*

$$\partial_s u(z) + J(z)\partial_t u(z) + S(z)u(z) = 0, \qquad u(0) = 0.$$

There exists a constant $0 < \delta < \varepsilon$ such that $u(z) \neq 0$ for all $z \in B_\delta \backslash \{0\}$. Moreover if $S \equiv 0$ then there exists a constant $0 < \delta < \varepsilon$ such that $Du(z) \neq 0$ for each $z \in B_\delta \backslash \{0\}$.

Proof. The first statement is immediate from Proposition 7.9. The second statement is obvious if $Du(0) \neq 0$, so assume that $Du(0) = 0$. Set $v := \partial_s u$. Then

$$\partial_s v + J(z)\partial_t v + (\partial_s J)(z)J(z)v = 0.$$

Now apply the first statement to v, noting that $Du(z) = 0$ if and only if $\partial_s v(z) = 0$. ∎

Definition 7.13. A function $u \in W^{1,p}(\mathbb{C}, \mathbb{C}^n)$ is said to *vanish to infinite order* at a point $z_0 \in \mathbb{C}$ if

$$\sup_{k\in\mathbb{Z}} \lim_{r\downarrow 0} \frac{\sup_{z\in B(z_0;r)} |u(z)|}{r^k} = 0.$$

If u is smooth then the set of points that u vanishes to infinite order is closed. Moreover if u is holomorphic then the set of such points is both open and closed.

Proposition 7.14 (The Principle of Analytic Continuation)**.** *Suppose $J : \mathbb{C} \times \mathbb{C}^n \to \mathsf{L}_{\mathbb{R}}(\mathbb{C}^n)$ is of class $W^{1,p}$ with $p > 2$, and suppose that $J(z,w)^2 = -\mathbb{1}$. Suppose that $S : \mathbb{C} \times \mathbb{C}^n \to \mathsf{L}_{\mathbb{R}}(\mathbb{C}^n)$ is of class $W^{1,p}$. Suppose u and v are two solutions of the equation*

$$\partial_s u + J(z,u)\partial_t u + S(z,u(z)) = 0,$$

defined on open subset $\Omega \subset \mathbb{C}$. Then the set of points in Ω such that $u - v$ vanishes to infinite order is both open and closed.

Proof. Set $w := u - v$ and define

$$T(z) := \left(\int_0^1 \frac{d}{dr}(S(z, u + r(v-u)dr \right) \partial_t v + \int_0^1 \frac{d}{dr} S(z, u + \tau(v-u),$$

and hence w satisfes

$$\partial_s w + \tilde{J}(z)\partial_r w + T(z)w = 0,$$

where $\tilde{J}(z) := J(z, u(z))$. The result now follows from the Carleman Similarity Principle, combined with the analogous statement about holomorphic maps. ■

We now prove that the set $C(u)$ of points where $\partial_s u$ vanishes of a solution of Floer's equations is discrete.

Lemma 7.15. *Suppose $J : \mathbb{C}^n \to \mathsf{L}_{\mathbb{R}}(\mathbb{C}^n)$ is of class C^k with $p > 2$, and suppose that $J(z)^2 = -\mathbb{1}$. Suppose that $X : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ is a vector field of class C^k. is of class $W^{1,p}$. Suppose $u : B_\varepsilon \to \mathbb{C}$ is a $W^{1,p}$-solution (and hence a C^k-solution by elliptic regularity) of the equation*

$$\partial_s u + J(u)\partial_t u - J(u)X(t, u(z)) = 0.$$

Then the set $C(u)$ of points $z = s + it$ in B_ε for which $\partial_s u(s,t) = 0$ is discrete.

Proof. Let $\phi_t : \Omega_t \to \mathbb{C}^n$ denote the local flow of $X(t, \cdot)$. The assertion of the lemma is local, so it sufficies to consider the case $u(s,t) \in \phi_t(\Omega_t)$ for each $z \in B_\varepsilon$. We wish to reduce the lemma to the case $X(t, \cdot) \equiv 0$. Set $v(s,t) := (\phi_t)^{-1}(u(s,t))$. Then

$$\partial_s u = D\phi_t(v)\partial_s v, \qquad \partial_t u - X(t,u) = D\phi_t(v)\partial_t v,$$

and hence if

$$\tilde{J}(t,z) := (D\phi_t(z))^{-1} J(\phi_t(z)) \circ D\phi_t(z)$$

then

$$\partial_s v + \tilde{J}(t,z)\partial_t v = 0.$$

Moreover $Dv(z) = 0$ if and only if $\partial_s u(z) = 0$. Finally Corollary 7.12 tells us that the critical points of v are discrete. ■

We will need one more preliminary result before embarking on the proof of Theorem 7.6.

Lemma 7.16. *Suppose $J : \mathbb{R} \times \mathbb{C}^n \to \mathsf{L}_{\mathbb{R}}(\mathbb{C}^n)$ is of class $W^{1,p}$ with $p > 2$, and suppose that $J(t,z)^2 = -\mathbb{1}$. Suppose $u, v \in C^k(B_\varepsilon, \mathbb{C}^n)$ satsify*

$$\partial_s u + J(t,u)\partial_t u = 0, \qquad \partial_s v + J(t,v)\partial_t v = 0,$$

$$u(0) = v(0), \qquad Du(0) \neq 0, \qquad Dv(0) \neq 0.$$

Suppose moreover that there exists a constant $0 < \delta < \varepsilon$ such that for all $(s,t) \in B_\delta$ there exists another point $(s',t) \in B_\varepsilon$ such that $u(s,t) = v(s',t)$. Then u and v coincide on B_ε.

Proof. Up to shrinking ε, we may assume that $N := v(B_\varepsilon)$ is a submanifold of $\mathbb{C}^n$. The implicit function theorem implies that the map $v^{-1} : N \to B_\varepsilon$ extends to a C^k map defined on a neighborhood of N. Moreover by assumption $u(B_\delta) \subset N$, and the composition $v^- \circ u : B_\delta \to B_\varepsilon$ is of the form

$$v^{-1} \circ u(s,t) = (\psi(s,t), t).$$

Differentiating the equation $u(s,t) = v(\psi(s,t), t)$ we obtain

$$\begin{aligned} 0 &= \partial_s u(s,t) + J(t,u)\partial_t u(s,t) \\ &= \partial_s v(\psi,t)\partial_s \psi + J(t, v(\psi,t))(\partial_s v(\psi,t)\partial_t \psi + \partial_t v(\psi,t)) \\ &= (\partial_s \psi - 1)\partial_s v(\psi,t) + \partial_t \psi \partial_t v(\psi,t). \end{aligned}$$

Since $\partial_s v(\psi,t)$ and $\partial_t v(\psi,t)$ are linearly independent, we see that $\partial_s \psi = 1$ and $\partial_t \psi = 0$.Thus $\psi(s,t) = s + s_0$ for some $s_0 \in \mathbb{R}$. Since both u and v fix the origin, we see that $s_0 = 0$. Thus on a small neighborhood fo 0, u and v coincide. By the principle of analytic continuation, in fact they coincide everywhere. This completes the proof. ∎

We can now finally complete the proof of Theorem 7.6

Proof of Theorem 7.6. We first reduce the proof to the case $X_H \equiv 0$. This is done via the same trick as the one employed in the proof of Lemma 7.15: consider the flow ϕ_H^t of X_H and replace u with $v(s,t) := (\phi_H^t)^{-1}(u(s,t))$. Then v satisfies the equation

$$\partial_s v + \tilde{J}(t,v)\partial_t v = 0,$$

where

$$\tilde{J}_t(x) := (D\phi_H^t)^{-1} \circ J(\phi_t(x)) \circ D\phi_t(x).$$

Moreover

$$\lim_{s \to \pm\infty} v(s,t) = q^\pm,$$

where $x^\pm(0) := q^\pm$. By definition $R(u) = R(v)$, and thus it suffices to show that $R(v)$ is open and dense. Suppose $R(v)$ is not open: then there exists $(s,t) \in R(v)$ such that there exists $(s_k, t_k) \to (s,t)$ with $(s_k, t_k) \in \mathbb{R}^2 \backslash R(v)$. I have a sneaking suspicion that no one is reading these notes (apart from me). So as a test: the first person to tell me in class that the moon is made of cheese gets 100% in the exam[3]. Since $(s,t) \in R(v)$, we see that for k sufficiently large we have $\partial_s v(s_k, t_k) \neq 0$ and $v(s_k, t_k) \neq q^\pm$. Thus the third condition in the definition of regular points must fail: there exists $s_k' \neq s_k$ such that $v(s_k, t_k) = v(s_k', t_k)$ for each $k \in \mathbb{Z}$. If s_k' is unbounded, passing to subsequence we may assume $s_k' \to \pm\infty$. This implies that $v(s,t) = q^\pm$, which contradicts the assertion that $(s,t) \in R(v)$. Thus s_k' is bounded, and hence we may assume $s_k' \to s'$. Then $v(s,t) = v(s', t)$, and since $(s,t) \in R(v)$, this implies that $s = s'$. Since both $s_k \to s$ and $s_k' \to s$, with $v(s_k, t_k) = v(s_k', t_k)$ for each k, it follows that $\partial_s v(s,t) = 0$, which contradicts the fact that $(s,t) \in R(v)$.

The argument that $R(v)$ is dense is more involved, and we refer the reader to [FHS96, p261-262]. Here we only outline the details:

1. If there exists $(s_0, t_0) \in (\mathbb{R} \times S^1) \backslash C(v)$ and $\varepsilon > 0$ such that $B((s_0,t_0); \varepsilon) \cap R(v) = \emptyset$ then there exists $s_1 \in \mathbb{R}$ and $0 < \delta < \varepsilon$ such that on $B((s_0,t_0); \delta)$ the conditions of Lemma 7.16 are satisfied by $v(s,t)$ and $v(s + s_1, t)$.

[3] Well done to C. Antony for passing the test. Everyone else FAILED.

2. Lemma 7.16 thus implies that $v(s,t) = v(s+s_1,t)$ on $B((s_0,t_0);\delta)$, and the principle of analytic continuation (Proposition 7.14) implies that $v(s,t) = v(s+s_1,t)$ for all $(s,t) \in \mathbb{R} \times S^1$.

3. This implies that v is constant: indeed for each $(s,t) \in \mathbb{R} \times S^1$,

$$v(s,t) = \lim_{k\to\pm\infty} v(s+ks_1,t) = q^{\pm}.$$

   But this implies that $u(s,t) = x^-(t) = x^+(t)$, which contradicts the assertion that $x^- \neq x^+$.

Obviously the hard work is in verifying (1). This we omit[4]. ■

[4] Since I am really tired...

CHAPTER 8

Compactness and gluing

Let us now fix once and for all an almost complex structure $J \in \mathcal{J}(Q, \omega)$ and a Hamiltonian $H \in C^\infty(S^1 \times Q)$ such that every element of $\mathcal{P}_1^\circ(H)$ is non-degenerate, and such that for any two orbits x^-, x^+ belonging to $\mathcal{P}_1^\circ(H)$ and any $u \in \mathcal{M}(x^-, x^+)$, the vertical derivative $D^v \bar{\partial}_{J,H}(u)$ is surjective. In other words, from now on we will always assume that all the moduli spaces are manifolds. The aim of this section is to discuss the compactness properties of the spaces $\mathcal{M}(x^-, x^+)$. Of course, as we have seen the space $\mathcal{M}(x^-, x^+)$ can never be compact if $x^- \neq x^+$, since it carries a free translation $\mathbb{R}$-action. However as we will see the space $\underline{\mathcal{M}}(x^-, x^+) := \mathcal{M}(x^-, x^+)/\mathbb{R}$ is sometimes compact.

Let us denote by $\mathcal{M}^\sharp$ the space of *all* gradient flow lines:

$$\mathcal{M}^\sharp := \left\{ u \in C^\infty(\mathbb{R} \times S^1, Q) \mid u(s, \cdot) \in \mathcal{L}Q, \ \forall\, s \in \mathbb{R}, \ \bar{\partial}_{J,H}(u) = 0 \right\}. \tag{8.1}$$

Definition 8.1. We define the *energy* $\mathbb{E} : \mathcal{M}^\sharp \to [0, +\infty]$ by

$$\mathbb{E}(u) := -\int_{-\infty}^{+\infty} \frac{d}{ds} \mathbb{A}_H(u(s, \cdot)) ds.$$

Note that

$$\begin{aligned}
\mathbb{E}(u) &= -\int_{-\infty}^{+\infty} d\mathbb{A}_H(u(s, \cdot)[\partial_s u(s, \cdot)] ds \\
&= -\int_{-\infty}^{+\infty} \langle\langle \nabla \mathbb{A}_H(u(s, \cdot)), \partial_s u \rangle\rangle_J ds \\
&= \int_{-\infty}^{+\infty} \langle\langle \partial_s u, \partial_s u \rangle\rangle ds \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \int_{S^1} \left(|\partial_s u|_J^2 + |\partial_t u - X_{H_t}(u)|_J^2 \right) ds.
\end{aligned}$$

In particular, this proves:

Lemma 8.2. *The energy $\mathbb{E} : \mathcal{M}^\sharp \to [0, +\infty]$ satisfies:*

1. *One has $\mathbb{E}(u) \geq 0$, with equality if and only if u is a constant gradient flow line, that is, $u(s, t) \equiv x(t)$ for some $x \in \mathcal{P}_1^\circ(H)$.*
2. *If $u \in \mathcal{M}(x^-, x^+)$ then*
$$\mathbb{E}(u) = \mathbb{A}_H(x^-) - \mathbb{A}_H(x^+). \tag{8.2}$$

Corollary 8.3. *If $\mathcal{M}(x^-, x^+) \neq \emptyset$ then $\mathbb{A}_H(x^-) \geq \mathbb{A}_H(x^+)$ and $\mathrm{CZ}(x^-) \geq \mathrm{CZ}(x^+)$. If in addition $x^- \neq x^+$ then $\mathbb{A}_H(x^-) > \mathbb{A}_H(x^+)$ and $\mathrm{CZ}(x^-) > \mathrm{CZ}(x^+)$.*

Proof. We need only prove the last statement, and this is simply the observation that if $x^- \neq x^+$ then the space $\mathcal{M}(x^-, x^+)$ is at least one-dimensional if non-empty, since it is invariant under the translation $\mathbb{R}$-action. ∎

The next result is the analogue of Theorem 1.20, in the Floer setting. We shall prove a more general version of Theorem 8.4 in Theorem 8.23 below.

Theorem 8.4 (*Baby compactness*)**.** *Suppose* $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+) + 1$. *Then the moduli space* $\underline{\mathcal{M}}(x^-, x^+)$ *is compact, and hence a finite set.*

It follows from Theorem 8.4 that the following definition makes sense.

Definition 8.5. Suppose $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+) + 1$. Then we define

$$n(x^-, x^+) := \#_2 \underline{\mathcal{M}}(x^-, x^+).$$

We can then mimic the Morse-theoretic definition (cf. Definition def:the boundary operator). We define the *Floer chain group* to be We then have a well defined operator

$$\mathrm{CF}_k(H) := \bigoplus_{x \in \mathcal{P}_1^\circ(H),\, \mathrm{CZ}(x) = k} \mathbb{Z}_2 \langle x \rangle ,$$

and we define the boundary operator

$$\partial = \partial_J : \mathrm{CF}_k(H) \to \mathrm{CF}_{k-1}(H)$$

by requiring that

$$\partial \left\langle x^- \right\rangle := \sum_{y \in \mathcal{P}_1^\circ(H),\, \mathrm{CZ}(x^+) = \mathrm{CZ}(x^-) - 1} n(x^-, x^+) \left\langle x^+ \right\rangle ,$$

and then extending by linearity. However just as in the Morse case, Theorem 8.4 is not sufficient to prove that $\partial \circ \partial = 0$, and hence $(\mathrm{CF}_*(H), \partial)$ is a chain complex. For this we will need a more sophisticated compactness theorem, which we will state and prove later in this chapter (see Theorem **??**). To begin with, we will study the space of all *finite energy* gradient flow lines, and show that this space is compact.

Definition 8.6. We define

$$\mathcal{M} := \left\{ u \in \mathcal{M}^\sharp \mid \mathbb{E}(u) < +\infty \right\}.$$

Our first result about the space $\mathcal{M}$ is the following, which is a consequence of the elliptic regularity results from Chapter 4.

Proposition 8.7. *On* $\mathcal{M}$ *the* C^0_{loc} *and the* C^∞_{loc}*-topologies coincide.*

Proof. Suppose $(u_k) \in \mathcal{M}$ converges in C^0_{loc} to some u. We already know that all the u_k and u are of class C^∞, but what we don't yet know is whether u_k converges to u in the C^∞_{loc} topology. By working in a chart, we may assume that $Q = \mathbb{R}^{2n}$. Fix $p > 2$. Let us abbreviate

$$J_k(s,t) := J(u_k(s,t)), \qquad f_k(s,t) := J(u_k(s,t)) X_{H_t}(u_k(s,t)).$$

Thus J_k converges to $J(u)$ and f_k converges to $J(u)X_H(u)$ in C^0_{loc}, and hence also in L^p_{loc}. Note that

$$0 = \partial_s(u_k - u) + J_k(\partial_t(u_k - u)) + f_k - f + (J(u) - J_k)\partial_t u. \tag{8.3}$$

Fix open domains $U \subset \Omega \subset \mathbb{R} \times S^1$ with compact closure, such that $\overline{U} \subset \Omega$. Since $J_k \to J$ in L^p_{loc}, there exists a constant $c_0 > 0$ such that $\|J_k\|_{L^p(\Omega)} \le c_0$. We now apply Theorem 4.19 to $u_k - u$, using (8.3), to obtain a constant $c > 0$ such that

$$\|u_k - u\|_{W^{1,p}(U)} \le c \left(\|f_k - f\|_{L^p(\Omega)} + \|u_k - u\|_{L^p(\Omega)} + \|J_k - J(u)\partial_t u\|_{L^p(\Omega)} \right). \tag{8.4}$$

Since we already know that u_k converges to u in L^p_{loc}, the right-hand side of (8.4) is infinitesimal, and thus we see that u_k converges to u in $W^{1,p}_{\mathrm{loc}}$. But now we can iterate this to see that u_k converges to u in $W^{1,l}_{\mathrm{loc}}$ for all $l \in \mathbb{N}$. By Rellich's Compactness Theorem it follows that u_k converges to u in C^∞_{loc} as required. ∎

The next result is the first key property of the space $\mathcal{M}$.

Theorem 8.8. *[No bubbling] The space $\mathcal{M}$ is compact in the C^∞_{loc}-topology.*

Remark 8.9. Theorem 8.8 is known as the "No bubbling Theorem" because, as we will see, the main content is the assertion that "bubbles" cannot form. In general the existence of bubbles are the obstruction to compactness in Floer theory. The main reason we assumed that the map $I_\omega : \pi_2(Q) \to \mathbb{R}$ from Definition 2.18 vanished was to exclude bubbling.

Remark 8.10. In fact, a slightly stronger result is true. If H is degenerate but the the set $\mathcal{P}_1^\circ(H)$ is finite then the space $\mathcal{M}$ is still compact (although in this case the spaces $\mathcal{M}(x^-, x^+)$ do not necessarily carry a manifold structure!). We will leave the verification of this additional statement to the reader; one simply needs to note that the proof only ever uses the fact that $\mathcal{P}_1^\circ(H)$ is finite.

For now we will assume Theorem 8.8 and proceed. The proof will be given later in this chapter. Observe that (8.2) implies that $\mathcal{M}(x^-, x^+) \subset \mathcal{M}$ for each pair x^-, x^+ in $\mathcal{P}_1^\circ(H)$. In fact, the converse holds.

Theorem 8.11. *The space $\mathcal{M}$ is precisely the union of the spaces $\mathcal{M}(x^-, x^+)$:*

$$\mathcal{M} = \bigcup_{x^-, x^+ \in \mathcal{P}_1^\circ(H)} \mathcal{M}(x^-, x^+).$$

Remark 8.12. Recall from Definition 4.4, the spaces $\mathcal{M}(x^-, x^+)$ were defined as subsets of a space $\mathcal{B}^{1,p}(x^-, x^+)$ of maps that satisfies a certain asymptotic decay condition (see (4.4)). Thus one consequence of Theorem 8.11 is that all finite energy gradient flow lines exhibit this asymptotic behaviour.

Remark 8.13. In fact, as with Remark 8.10, Theorem 8.11 only needs us to assume that the set $\mathcal{P}_1^\circ(H)$ is finite. Again, we leave this to the conscientious reader to check.

We will now prove Theorem 8.11, assuming Theorem 8.8. The first step is the following cute little lemma.

Lemma 8.14. *Let (X, d) be a complete metric space and $h : X \to (0, +\infty)$ a continuous function. For every $x_0 \in X$ and $\varepsilon_0 > 0$ there exists $x_1 \in B(x_0; 2\varepsilon_0)$ and $0 < \varepsilon_1 \le \varepsilon_0$ such that*

$$h(x_1)\varepsilon_1 > h(x_0)\varepsilon_0,$$

and such that

$$h(y) \le 2h(x_1), \qquad \text{for all } y \in B(x_1; \varepsilon_1).$$

Proof. If $h(x) \le 2h(x_0)$ for all $x \in B(x_0; \varepsilon_0)$ we are done with $x_1 := x_0$ and $\varepsilon_1 := \varepsilon_0$. Otherwise there exists $y_1 \in B(x_0; \varepsilon_0)$ such that $h(y_1) > 2h(x_0)$. Set $\delta_1 := \frac{1}{2}\varepsilon_0$, so that $\delta_1 h(y_1) > \varepsilon_0 h(x_0)$. Again, if $h(x) \le 2h(y_1)$ for all $x \in B(y_1; \delta_1)$ we are done with $x_1 := y_1$ and $\varepsilon_1 := \delta_1$. If not then we continue: select $y_2 \in B(y_1; \delta_1)$ such that $h(y_2) > 2h(y_1)$, and set $\delta_2 := \frac{1}{2}\delta_1$, and argue as before. We claim that we must be done at some finite stage. Indeed, if not then since the sequence $\delta_k = 2^{-k}\varepsilon_0 \to 0$ and X is complete, the sequence y_k converges to some y. Since $\delta_k h(y_k) \to 0$ we obtain $h(y) = 0$, which contradicts the assertion that $h > 0$. ∎

The next step is the following lemma.

Lemma 8.15. *Suppose $u \in \mathcal{M}$. Then there exists a constant $C = C(u)$ such that*

$$\sup_{(s,t) \in \mathbb{R} \times S^1} |\nabla u(s, t)|_J \le C. \tag{8.5}$$

Remark 8.16. In fact, later on we will improve Lemma 8.15 to show that the assumption that I_ω vanishes implies that one can even choose the constant C appearing in (8.9) to be *independent* of u. See Lemma 8.18 below.

Proof of Lemma 8.15. Assume for contradiction there exists a sequence $(s_k, t_k) \in \mathbb{R} \times S^1$ such that $R_k := |\nabla u(s_k, t_k)|_J \to +\infty$. Let ε_k be a sequence such that $\varepsilon_k \to 0$ and $\varepsilon_k R_k \to +\infty$. By Lemma 8.14, we may assume that

$$|\nabla u(s,t)|_J < 2R_k, \qquad \text{for all } |(s,t) - (s_k, t_k)| < \varepsilon_k.$$

For $(s,t) \in \mathbb{R}^2 \cong \mathbb{C}$, define

$$v_k(s,t) := u_k(s_k + s/R_k, t_k + t/R_k).$$

Thus $\|v_k(0,0)\|_J = 1$ and

$$|\nabla v_k(s,t)|_J \leq 2, \qquad \text{for all } (s,t) \in B(0; \varepsilon_k R_k). \tag{8.6}$$

Note that

$$\partial_s v_k + J(v_k)\partial_t v_k = \frac{1}{R_k} J(v_k) X_{H_{t_k + t/R_k}}(v_k).$$

From (8.6) we see that (v_k) is equicontinuous. Compactness of Q implies that (v_k) is totally bounded, and hence by Arzela-Ascoli theorem, after passing to a subsequence we may assume that v_k converges in C^0_{loc} to some $v : \mathbb{C} \to Q$. Moreover v satisfies

$$\partial_s v + J(v)\partial_t v = 0, \tag{8.7}$$

and hence elliptic regularity implies that v is smooth. Moreover the argument from Proposition 8.7 shows that v_k converges to v in C^∞_{loc}. In particular, $\|\nabla v(0,0)\|_J = 1$. However we also have

$$\begin{aligned}
\int_{\mathbb{C}} |\partial_s v|_J^2 ds dt &= \lim_{l \to \infty} \int_{B(0;\varepsilon_l R_l)} |\partial_s v|_J^2 ds dt \\
&= \lim_{l \to \infty} \lim_{k \to \infty} \int_{B(0;\varepsilon_l R_l)} |\partial_s v_k|_J^2 ds dt \\
&= \lim_{l \to \infty} \lim_{k \to \infty} \int_{B((s_l,t_l);\varepsilon_l R_l / R_k)} |\partial_s u|_J^2 ds dt \\
&= 0,
\end{aligned}$$

since $|\nabla u|_J$ is bounded. This implies that $\partial_s v \equiv 0$, and thus from (8.7), $\partial_t v \equiv 0$. This contradicts $|\nabla v(0,0)|_J = 1$. ∎

Next, we prove:

Lemma 8.17. *Suppose $u \in \mathcal{M}$ and $(s_k) \subset \mathbb{R}$ converges to $+\infty$. Then there exists $x \in \mathcal{P}_1^\circ(H)$ such that $(u(s_k, \cdot))$ converges in $C^\infty(S^1, Q)$ to x.*

Proof. Lemma 8.15 implies that the sequence $(u(\cdot + s_k, \cdot))$ is equicontinuous. Compactness of Q implies it is totally bounded, and hence the Arzela-Ascoli implies, up to passing to a subsequence, that $(u(\cdot + s_k, \cdot))$ converges in C^0_{loc} to some v. Moreover elliptic regularity implies that $v \in C^\infty(\mathbb{R} \times S^1, Q)$ and $\bar{\partial}_{J,H}(v) = 0$. Then Proposition 8.7 implies that the convergence is in C^∞_{loc}, and hence

$$\begin{aligned}
\mathbb{E}(v) &= \int_{\mathbb{R}\times S^1} |\partial_s v|_J^2 ds dt \\
&= \lim_{R\to+\infty} \int_{[-R,R]\times S^1} |\partial_s v|_J^2 ds dt \\
&= \lim_{R\to+\infty} \lim_{k\to\infty} \int_{[-R,R]\times S^1} |\partial_s u(s+s_k,t)|_J^2 ds dt \\
&= 0,
\end{aligned}$$

since $|\nabla u\|_J$ is square integrable. Thus by Lemma 8.2, $v(s,t) \equiv x(t)$ for some $x \in \mathcal{P}_1^\circ(H)$, which is what we wanted to prove. ■

We can now complete the proof of Theorem 8.11.

Proof of Theorem 8.11. It suffices to show that if $u \in \mathcal{M}$ and (s_k) and (σ_k) are two sequences that both converge to $+\infty$, such that $u(s_k,\cdot) \to x_1$ and $u(\sigma_k,\cdot) \to x_2$, for two orbits x, y in $\mathcal{P}_1^\circ(H)$, then $x_1 = x_2$. Let d denote the metric on $\mathcal{L}Q$ defined by

$$d(x,y) := \sup_{t\in S^1} d(x(t), y(t)), \tag{8.8}$$

where d is some compatible metric on M. Since $\mathcal{P}_1^\circ(H)$ is finite, there exists $\varepsilon > 0$ such that the sets $B(x;\varepsilon)$ are mutually disjoint for $x \in \mathcal{P}_1^\circ(H)$. If $x_1 \neq x_2$ then using the sequences (s_k) and (σ_k), we can find sequences (a_k) and (b_k) such that $a_k, b_k \to +\infty$ and such that $a_k < b_k < a_{k+1}$, with $u(a_k,\cdot) \to x_1$ and $u(b_k,\cdot) \to x_2$. For all large k we may assume that $u(a_k,\cdot) \in B(x_1;\varepsilon)$ and $u(b_k,\cdot) \in B(x_2;\varepsilon)$. Since these sets are pairwise disjoint, it follows that we can find a sequence $a_k < c_k < b_k$ such that

$$u_k(c_k,\cdot) \notin \bigcup_{x\in\mathcal{P}_1^\circ(H)} B(x;\varepsilon), \qquad \text{for all } k \in \mathbb{N}.$$

Since $c_k \to +\infty$, this contradicts Lemma 8.17. ■

This completes the proof of Theorem 8.11. We will now prove the No bubbling Theorem, Theorem 8.8. The argument is an elaboration of the one from Lemma 8.15. Namely, let us first show:

Lemma 8.18. *Then there exists a constant $C > 0$ such that for all $u \in \mathcal{M}$, one has*

$$\sup_{(s,t)\in\mathbb{R}\times S^1} |\nabla u(s,t)|_J \leq C. \tag{8.9}$$

Proof. We apply exactly the same argument as in the proof of Lemma 8.15, only starting with a sequence (s_k, t_k) such that $R_k := |\nabla u_k(s_k,t_k)|_J \to +\infty$. Again we obtain a non-constant map $v \in C^\infty(\mathbb{C}, Q)$ satisfying $\partial s v + J(v)\partial_t v = 0$; the only difference is that this time we cannot conclude that $\mathbb{E}(v) = 0$. However we certainly have $\mathbb{E}(v) < +\infty$; for instance we have

$$\mathbb{E}(v) \leq \sup_{k\in\mathbb{N}} \mathbb{E}(u_k) \leq \max_{x^-,x^+\in\mathcal{P}_1^\circ(H)} \mathbb{A}_H(x^-) - \mathbb{A}_H(x^+),$$

where the second inequality uses Lemma 8.2. The proof of Lemma 8.18, and hence of Theorem 8.8, is thus completed by the following corollary. ■

Lemma 8.19 (No bubbling)**.** *Let (Q,ω) be a closed symplectic manifold such that $I_\omega : \pi_2(Q) \to \mathbb{R}$ is zero. Fix $J \in \mathcal{J}(Q,\omega)$. Then there exist no non-constant J-holomorphic maps $v : \mathbb{C} \to Q$ with finite energy.*

Proof. Suppose v is such a map. We build a new map $w : S^2 \to \mathbb{Q}$ such that $I_\omega([w]) \neq 0$. First define a new map $u : \mathbb{R} \times S^1 \to Q$ by

$$u(s,t) := v(e^{2\pi(s+it)}).$$

Then u is J-holomorphic and $\mathbb{E}(u) < +\infty$. The argument from Lemma 8.17 (applied with $H = 0$) tells us that there exists two points $q_-, q_+ \in Q$ such that

$$\lim_{s\to\pm\infty} u(s,t) \equiv q^{\pm}$$

(since $\mathcal{P}_1^\circ(H = 0)$ is the set of constant maps). Thus if we consider a smooth map $\zeta_k : \mathbb{D} \to Q$ such that $\zeta_k(e^{2\pi it}) = u(k,t)$ then

$$\lim_{k\to+\infty} \int_{\mathbb{D}} \zeta_k^* \omega = 0. \tag{8.10}$$

We can glue $v|_{B(0;e^{2\pi k}}$ and ζ_k together to obtain a map $w_k : S^2 = \mathbb{C} \cup \{+\infty\} \to Q$ which is smooth away from $\partial B(0; e^{2\pi k})$ and has one-sided derivative in the direction normal to this circle. Then

$$\begin{aligned} 0 &= I_\omega[w_k] \\ &= \int_{B(0;e^{2\pi k})} v^*\omega - \int_{\mathbb{D}} \zeta_k^*\omega \end{aligned}$$

Thus from (8.10), we see that in fact $\int_{\mathbb{C}} v^*\omega = 0$. To complete the proof one uses the following simple exercise.

Exercise 8.20. Suppose $v \in C^\infty(\mathbb{C}, Q)$ is J-holomorphic for some $J \in \mathcal{J}(Q,\omega)$. Then

$$\int_{\mathbb{C}} v^*\omega = \int_{\mathbb{C}} |\partial_s v|_J^2.$$

■

Let us now move to studying the quotient spaces $\underline{\mathcal{M}}(x^-, x^+)$ in more detail. We equip $\underline{\mathcal{M}}(x^-, x^+)$ with quotient topology. Let us denote by $\underline{u}$ an element of $\underline{\mathcal{M}}(x^-, x^+)$. Thus given a sequence $\underline{u}_k \in \underline{\mathcal{M}}(x^-, x^+)$, we have

$$\underline{u}_k \to \underline{u} \in \underline{\mathcal{M}}(x^-, x^+) \qquad \Longleftrightarrow \qquad \begin{cases} \exists\, u_k \in \underline{u}_k \text{ and } u \in \underline{u}, & \exists\,(s_k) \subset \mathbb{R}, \\ u_k(\cdot + s_k, \cdot) \to u, & \text{in } C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1, Q). \end{cases}$$

Proposition 8.21. *Suppose x^- and x^+ are two distinct elements of $\mathcal{P}_1^\circ(H)$. Suppose $(\underline{u}_k) \subset \underline{\mathcal{M}}(x^-, x^+)$ has two subsequences which converge to elements $\underline{u} \in \underline{\mathcal{M}}(x^-, y)$ and $\underline{v} \in \underline{\mathcal{M}}(x^-, z)$. Then $\underline{u} = \underline{v}$. In particular, $y = z$.*

Proof. Up to passing to a subsequence, there exists a sequence $(u_k) \subset \mathcal{M}(x^-, x^+)$ and two sequences $(s_k), (\sigma_k) \subset \mathbb{R}$ such that

$$\lim_{k\to+\infty} u_k(\cdot + s_k, \cdot) = u \in \mathcal{M}(x^-, y), \qquad \lim_{k\to+\infty} u_k(\cdot + \sigma_k, \cdot) = v \in \mathcal{M}(x^-, z).$$

Select $a, \varepsilon > 0$ such that

$$\mathbb{A}_H(x^-) - \varepsilon > a > \max\left\{\mathbb{A}_H(x^+), \mathbb{A}_H(y), \mathbb{A}_H(z)\right\}.$$

Suppose that the sequence $|s_k - \sigma_k|$ is not bounded. Then without loss of generality we may assume that $s_k = 0$ and $\sigma_k \to +\infty$. Since $v \in \mathcal{M}(x^-, z)$, there exists $s_0 \in \mathbb{R}$ such that

$$s \le s_0 \qquad \Rightarrow \qquad \mathbb{A}_H(v(s, \cdot)) > \mathbb{A}_H(x^-) - \varepsilon.$$

Since $u_k(s_0 + \sigma_k, \cdot)$ converges to $v(s_0, \cdot)$, for all k sufficiently large, one also has

$$\mathbb{A}_H(u_k(s_0 + \sigma_k, \cdot)) > \mathbb{A}_H(x^-). \tag{8.11}$$

However since $u \in \mathcal{M}(x^-, y)$, there exists $s_1 \in \mathbb{R}$ such that

$$s \ge s_1 \qquad \Rightarrow \qquad \mathbb{A}_H(u(s, \cdot)) < a.$$

Since $s_k = 0$ by assumption, we have that for all k sufficiently large,

$$\mathbb{A}_H(u_k(s_1, \cdot)) \le a + \varepsilon. \tag{8.12}$$

Since $\sigma_k \to +\infty$, for k sufficiently large one has $s_0 + \sigma_k > s_1$, and for such k, (8.11) and (8.12) are mutually incompatible. Thus $|s_k - \sigma_k|$ is necessarily a bounded sequence, and hence after passing to a subsequence we may assume $\sigma_k - s_k \to s_0$. Since for any fixed $s \in \mathbb{R}$ the sequence $s + \sigma_k - s_k$ is contained in a compact subset of $\mathbb{R}$, we see that $u_k(s + \sigma_k, \cdot)$ converges to both $v(s, \cdot)$ and $u(s + s_0, \cdot)$. This completes the proof. ∎

With these preparations complete, we can now mimic the construction from Definition 1.24 in the Floer setting.

Definition 8.22. A sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}(x^-, x^+)$ is said to *converge up to breaking* if there exist

1. elements $x = x^0, x^1, \dots, x^m = x^+$ in $\mathcal{P}_1^\circ(H)$,
2. flow lines $u^j \in \mathcal{M}(x^{j-1}, x^j)$ for $1 \le j \le m$,
3. sequences $(s_k^j)_{k\in\mathbb{N}}$ for $1 \le j \le m$ with $s_k^{j-1} < s_k^j$ and with $s_k^j - s_k^{j-1} \to +\infty$ for each $k \in \mathbb{N}$ and each $1 \le j \le m$,

with the following property: For any compact interval $I \subset \mathbb{R}$, after passing to a subsequence, the sequence $u_k(\cdot + s_k^j, \cdot)$ converges in C^∞_{loc} to $u^j(\cdot)$. In this case we say that (u_k) converges to the *broken gradient flow line* $(u^1, \dots, u^m)$ and we write

$$u_k \rightsquigarrow (u^1, \dots, u^m).$$

Similarly we say that a sequence $(\underline{u}_k) \subset \underline{\mathcal{M}}(x^-, x^+)$ *converges to the broken gradient flow line* $(\underline{u}^1, \dots, \underline{u}^m)$, written

$$\underline{u}_k \rightsquigarrow (\underline{u}^1, \dots, \underline{u}^m),$$

if there exist representatives $u_k \in \underline{u}_k$ and $u^j \in \underline{u}^j$ such that $u_k \rightsquigarrow (u^1, \dots, u^m)$.

Note that if there exists a sequence $(\underline{u}_k) \subset \underline{\mathcal{M}}(x^-, x^+)$ that converges to a broken flow line $(\underline{u}^1, \dots, \underline{u}^m)$ then necessarily $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+) \ge m + 1$. Indeed, one has $\underline{u}^j \in \underline{\mathcal{M}}(x^{j-1}, x^j)$ for some critical points $x = x^0, \dots, x^m = y$, and then $\mathrm{CZ}(x^{j-1}) - \mathrm{CZ}(x^j) \ge 1$ for each $1 \le j \le m$ (cf. Corollary 8.3). We now state the full compactness result for Floer flow lines; contrast this to Theorem 1.26.

Theorem 8.23 (*Compactness*)**.** *Suppose x^-, x^+ are elements of $\mathcal{P}_1^\circ(H)$ which satisfy*

$$\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+) + m + 1$$

for some $m \geq 0$. Then $\underline{\mathcal{M}}(x^-, x^+)$ is compact up to m-fold breaking *in the following sense: suppose $(\underline{u}_k) \subset \underline{\mathcal{M}}(x^-, x^+)$ has no convergent subsequence. Then there exists a broken gradient trajectory $(\underline{u}^1, \dots, \underline{u}^l)$ for some $l \leq m$ such that after passing to a subsequence, $\underline{u}_k \rightsquigarrow (\underline{u}^1, \dots, \underline{u}^l)$. In particular, if $m = 0$ then every sequence in $\underline{\mathcal{M}}(x^-, x^+)$ has a convergent subsequence.*

Proof. Recall from the proof of Theorem 8.11 that we can choose $\varepsilon > 0$ such that the balls $B(x; \varepsilon)$ for $x \in \mathcal{P}_1^\circ(H)$ are all disjoint, where the balls are taken with respect to the metric (8.8). Suppose $(u_k) \subset \mathcal{M}(x^-, x^+)$. Since the hypotheses imply that $x^- \neq x^+$, there exists a well defined finite number

$$s_k^1 := \inf \left\{ s \in \mathbb{R} \mid d(u_k(s, \cdot), x^-) > \varepsilon \right\}. \tag{8.13}$$

Since $\mathcal{M}$ is compact, up to passing to a subsequence we may assume that $u_k(\cdot + s_k^1, \cdot)$ converges to some $u^1 \in \mathcal{M}$. It follows from (8.13) that $u^1(s, \cdot) \in \bar{B}(x^-; \varepsilon)$ for all $s \leq 0$, and $u^1(0, \cdot) \in \partial B(x^-; \varepsilon)$. Thus $u^1 \in \mathcal{M}(x^-, x^1)$ for some $x^1 \in \mathcal{P}_1^\circ(H)$. If $x^1 = x^+$, we are done (with $l = 1$).

Let us now assume for some $1 \leq p \leq m$ we have found sequence (s_k^j) for $1 \leq j \leq p$ and flow lines $u^j \in \mathcal{M}(x^{j-1}, x^j)$ for $1 \leq j \leq p$ such that $x^p \neq x^+$ and such that $u_k(\cdot + s_k^j, \cdot)$ converges to u^j in C^∞_{loc} for each $1 \leq j \leq p$. Since $u^p \in \mathcal{M}(x^{p-1}, x^p)$, there exists $\bar{s} \in \mathbb{R}$ such that for all $s \geq \bar{s}$, one has $u^k(s, \cdot) \in B(x^p; \varepsilon)$. Thus for all k sufficiently large, we have $u_k(s_k^p + \bar{s}, \cdot) \in B(x^p; \varepsilon)$. But since by assumption $x^p \neq x^+$ and $u_k \in \mathcal{M}(x^-, x^+)$, there exists a well-defined finite number

$$s_k^{p+1} := \sup \left\{ s \geq s_k^p + \bar{s} \mid u_k(\sigma, \cdot) \in B(x^p; \varepsilon) \text{ for all } \sigma \in [s_k^p, +\bar{s}, s] \right\}. \tag{8.14}$$

Compactness of $\mathcal{M}$ implies that, up to passing to a subsequence, $u_k(\cdot + s_k^{p+1}, \cdot)$ converges to some $u^{p+1} \in \mathcal{M}$. Note that necessarily $s_k^{p+1} - s_k^p \to +\infty$. Indeed, if not then there exists a compact subset of $\mathbb{R}$ containing the interval $[\bar{s}, s_k^{p+1} - s_k^p]$ for all k. For each $s \in [\bar{s}, s_k^{p+1} - s_k^p]$ we have $u_k(s + s_k^p, \cdot) \in B(x^p; \varepsilon)$, and since we have C^∞_{loc}-convergence, this implies $u_k(s_k^{p+1}, \cdot) \in B(x^p; \varepsilon)$, which contradicts (8.14).

To complete the proof we must show there exists an orbit $x^{p+1} \neq x^p$ such that $u^{p+1} \in \mathcal{M}(x^p, x^{p+1})$. Fix $s < 0$. Then for k sufficiently large, we have

$$s_k^p + \bar{s} < s_k^{p+1} + s < s_k^{p+1},$$

and thus $u_k(s_k^{p+1} + s, \cdot) \in B(x^p; \varepsilon)$ by (8.14). Thus we have

$$u^{p+1}(s, \cdot) \in B(x^p; \varepsilon), \qquad \text{for all } s < 0.$$

Moreover $u^{p+1}(0, \cdot) \in \partial B(x^p; \varepsilon)$, and hence u^{p+1} exits the ball $B(x^p; \varepsilon)$. The result follows. ∎

We now move onto the converse of this result. The reader should contrast the next result with Theorem 1.27. Whilst essentially the complete analogue of that result is true, we will content ourselves with only stating what we need in order to deduce that $\partial \circ \partial = 0$.

Theorem 8.24 (*Gluing*)**.** *Suppose x^-, x^0, x^+ are elements of $\mathcal{P}_1^\circ(H)$ such that*

$$\mathrm{CZ}(x^-) = \mathrm{CZ}(x^0) + 1 = \mathrm{CZ}(x^+) + 2.$$

Suppose that $(\underline{u}, \underline{v}) \in \underline{M}(x^-, x^0) \times \underline{\mathcal{M}}(x^0, x^+)$. Then there exists $\rho_0 > 0$ and a differentiable map

$$\psi : [\rho_0, +\infty) \to \mathcal{M}(x^-, x^+)$$

such that the induced map

$$\Psi : [\rho_0, +\infty) \to \underline{\mathcal{M}}(x^-, x^+)$$

is an embedding with the following two properties:

1. *$\Psi(\rho)$ converges to the broken flow line $(\underline{u}, \underline{v})$ as $\rho \to +\infty$,*
2. *if $\underline{w}_k \in \underline{\mathcal{M}}(x^-, x^+)$ converges to the broken flow line $(\underline{u}, \underline{v})$ as $k \to +\infty$ then for all k sufficiently large there exists ρ_k such that $\underline{w}_k = \Psi(\rho_k)$.*

We conclude this section with two simple exercises.

Exercise 8.25. Prove Theorem 8.24. *Hint: In the unlikely event that you find this challenging, consult [AD10, p280324 and p433468].* NB: The fact that I am setting this as an exercise has got absolutely nothing[5] to do with the fact that I *really* don't want to have to lecture this proof.

Exercise 8.26. Use Theorem 8.24 to prove that $\partial \circ \partial = 0$ in Floer homology. *Hint: The proof is identical to Corollary 1.31.*

[5] This is a lie.

CHAPTER 9

Invariance of Floer homology

In this chapter we will show that the Floer homology groups $\mathrm{HF}_*(H,J) := \mathrm{H}_*(\mathrm{CF}_*(H), \partial_J)$ are independent of both the choice of non-degenerate Hamiltonian H and the almost complex structure J. We then discuss briefly how to construct the Floer complex in the *Morse-Bott* setting. This allows us to define $\mathrm{HF}_*(H,J)$ for Hamiltonians such that the action functional $\mathbb{A}_H$ is Morse-Bott (cf. Definition 1.42) rather than Morse, and again the Floer homology is independent of such H. It is easy to see that $H \equiv 0$ is Morse-Bott, and moreover it follows essentially from the definition that $\mathrm{HF}_*(0,J) \cong \mathrm{H}_{n+*}(Q;\mathbb{Z}_2)$. In conclusion we obtain that $\mathrm{HF}_*(H,J) \cong \mathrm{H}_{n+*}(Q;\mathbb{Z}_2)$ for all non-degenerate Hamiltonians, which completes the proof of Theorem 2.25.

Definition 9.1. Let us denote by $\mathcal{HJ}_{\mathrm{reg}} \subset C^\infty(S^1 \times Q) \times \mathcal{J}(Q,\omega)$ the set of pairs (H,J) such that every element of $\mathcal{P}_1^\circ(H)$ is non-degenerate, and such that for any two orbits x^-, x^+ belonging to $\mathcal{P}_1^\circ(H)$ and any $u \in \mathcal{M}(x^-,x^+)$, the vertical derivative $D^v\bar{\partial}_{J,H}(u)$ is surjective. We call such a pair (H,J) a *regular pair*.

So far we have constructed the Floer complex $\mathrm{HF}_*(H,J)$ for any regular pair. Since both $C^\infty(S^1 \times Q)$ and $\mathcal{J}(Q,\omega)$ are path connected, if (H^-,J^-) and (H^+,J^+) are two regular pairs then we can find a path in $C^\infty(S^1 \times Q) \times \mathcal{J}(Q,\omega)$ connecting them. In general however such a path will not belong to $\mathcal{HJ}_{\mathrm{reg}}$. We will be interested in paths of a certain form.

Definition 9.2. Suppose $(H^\pm,J^\pm)$ are two regular pairs. An *aymptotically constant* path connecting (H^-,J^-) to (H^+,J^+) is a smooth path

$$\chi : \mathbb{R} \to C^\infty(S^1 \times Q) \times \mathcal{J}(Q,\omega), \qquad \chi(s) = (H^s,J^s)$$

such that there exists $T > 0$ such that

$$\chi(s) = \begin{cases} (H^-,J^-), & s \le -T, \\ (H^+,J^+), & s \ge T. \end{cases} \tag{9.1}$$

In Definition 9.10 below we will introduce the notion of a *regular* asymptotically constant path. Here is the first main result of this chapter.

Theorem 9.3 (*Continuation maps*)**.** *Suppose $(H^\pm,J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose χ is a regular asymptotically constant path connecting (H^-,J^-) to (H^+,J^+). Then there is a well defined chain map*

$$\Phi_\chi : \mathrm{CF}_*(H^-) \to \mathrm{CF}_*(H^+), \qquad \Phi_\chi \circ \partial_{J^-} = \partial_{J^+} \circ \Phi_\chi,$$

inducing a map

$$\phi_\chi : \mathrm{HF}_*(H^-,J^-) \to \mathrm{HF}_*(H^+,J^+).$$

Moreover if (H^0,J^0) is another regular pair, and χ^0 and χ^1 are regular asymptotically constant paths connecting (H^-,J^-) to (H^0,J^0) and (H^0,J^0) to (H^+,J^+) respectively, then the induced maps satisfy

$$\phi_\chi = \phi_{\chi^0} \circ \phi_{\chi^1}. \tag{9.2}$$

Finally, if $(H^-,J^-) = (H^+,J^+)$ and χ is a constant path then this path is regular and the chain map Φ_χ is the identity.

The next corollary is immediate from Theorem 9.3.

Corollary 9.4 (*Invariance of Floer homology*)**.** *Suppose $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs.*

1. *For any regular asymptotically constant path χ connecting (H^-, J^-) to (H^+, J^+), the induced map $\phi_\chi : \mathrm{HF}_*(H^-, J^-) \to \mathrm{HF}_*(H^+, J^+)$ depends only on the pair $(H^\pm, J^\pm)$ and not on the path χ.*

2. *The map $\phi_\chi : \mathrm{HF}_*(H^-, J^-) \to \mathrm{HF}_*(H^+, J^+)$ is an isomorphism.*

Exercise 9.5. Prove Corollary 9.4 (assuming Theorem 9.3!).

Remark 9.6. In fact, a slightly stronger result is true. If $H^- = H^+ = H$ and $\chi(s) = (H, J^s)$ only changes the almost complex structure J then the chain map $\Phi_\chi : \mathrm{CF}_*(H) \to \mathrm{CF}_*(H)$ is an *isomorphism of chain complexes*. Explicitly this means that the matrix representing the chain map Φ_χ is upper triangular with 1's on the diagonal. In other words. More precisely, the map has the form

$$\Phi_\chi\langle x\rangle = \sum_{y \in \mathcal{P}_1^\circ(H), \mathrm{CZ}(x) = \mathrm{CZ}(y)} n_\chi(x, y)\langle y\rangle,$$

where

$$n_\chi(x, x) = 1, \qquad n_\chi(x, y) = 0 \quad \text{if} \quad \mathbb{A}_H(x) \le \mathbb{A}_H(y) \quad \text{and} \quad x \ne y. \tag{9.3}$$

Thus whilst at the level of homology the Floer complex is independent of both the Hamiltonian and the almost complex structure, the Floer complex is independent (up to an isomorphism of chain complexes) of the almost complex structure *at the chain level*. This explains why in our notation we will often write $\mathrm{HF}_*(H)$ and omit the J, despite the fact that $\mathrm{HF}_*(H, J)$ is independent of both H and J. This will be particular important in the non-compact setting below. For instance, as we will see, in the case $Q = T^*B$, it is no longer true that $HF_*(H, J)$ is independent of H, but it is still that true that $\mathrm{HF}_*(H, J)$ is independent of J.

Suppose now that $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose χ is an asymptotically constant path connecting (H^-, J^-) to (H^+, J^+). Fix $x^- \in \mathcal{P}_1^\circ(H^-)$ and $x^+ \in \mathcal{P}_1^\circ(H^+)$. We define the Banach bundle $\mathcal{E}^p \to \mathcal{B}^{1,p}(x^-, x^+)$ in exactly the same way as before (cf. Definition 4.4.)

Definition 9.7. We define a section

$$\bar\partial_\chi : \mathcal{B}^{1,p}(x^-, x^+) \to \mathcal{E}^p$$

by setting

$$\bar\partial_\chi(u) := \partial_s u + J^s(u)(\partial_t u) - J^s(u) X_{H_t^s}(u).$$

We write $\mathcal{N}_\chi(x^-, x^+)$ for the zero set of $\bar\partial_\chi$.

The next two results are the key to defining the chain map Φ_χ.

Theorem 9.8. *Suppose that $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose χ is an asymptotically constant path connecting (H^-, J^-) to (H^+, J^+). Then vertical derivative $D^v\bar\partial_\chi(u)$ at a zero $u \in \mathcal{N}_\chi(x^-, x^+)$ is a Fredholm operator of index $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$.*

The proof of Theorem 9.8 is essentially identical to the proof in the s-independent case (cf. Theorem 6.24). Thus we will say no more about it. The conscientious reader is however invited to:

Exercise 9.9. Prove Theorem 9.8.

Definition 9.10. Let us say that an asymptotically constant path χ is an asymptotically constant path connecting (H^-, J^-) to (H^+, J^+) is *regular* if for all $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$, and all $u \in \mathcal{N}_\chi(x^-, x^+)$, the vertical derivative $D^v\bar{\partial}_\chi(u)$ is surjective.

It follows from Theorem 9.8 that if χ is a regular asymptotically constant path connecting (H^-, J^-) to (H^+, J^+) then the moduli space $\mathcal{N}_{\tilde{\chi}}(x^-, x^+)$ are manifolds of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$. Note that the moduli spaces $\mathcal{N}_\chi(x^-, x^+)$ are *not* invariant under a translation $\mathbb{R}$-action, and hence it doesn't make sense to try and form the quotient spaces $\mathcal{N}_\chi(x^-, x^+)/\mathbb{R}$. This explains why the chain map Φ_χ has degree 0, whereas the boundary operator had degree -1.

Theorem 9.11. *Suppose now that $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose χ is an asymptotically constant path connecting (H^-, J^-) to (H^+, J^+). For any $\varepsilon > 0$ there exists a regular asymptotically constant path $\tilde{\chi}(s) = (\tilde{H}^s, J^s)$ connecting (H^-, J^-) to (H^+, J^+) such that $\|H^s - \tilde{H}^s\|_{C^\infty(S^1 \times Q)} < \varepsilon$ for all $s \in \mathbb{R}$.*

Remark 9.12. The proof of Theorem 9.11 is actually much easier than the proof oin the s-independent case (cf. Theorem 7.2). Why is this? The strategy of the proof if exactly the same as before, and just as in (7.3), one is led to needing to prove that if $\hat{w} \in L^q(u^*TQ)$ satisfies

$$\int_{\mathbb{R}}\int_{S^1} d\hat{H}^s_t(u)[\hat{w}]dtds = 0$$

for all Hamiltonians $\hat{H}^s_t$ then $\hat{w}$ is identically zero. But this time we are allowed to take $\hat{H}$ to depend on s! Specifically, $\hat{H}$ is required to be a tangent vector to an asymptotically constant path H^s connecting H^- and H^+. This makes the argument much easier: if $\hat{w}(s_0, t_0) \neq 0$ then we may assume that $\hat{w}(s,t) > 0$ for all (s,t) near (s_0, t_0) (since elliptic regularity tells us that $\hat{w}$ is continuous). One can then choose a function $\hat{H}$ supported in a neighborhood of (s_0, t_0) such that

$$\int_{\mathbb{R}}\int_{S^1} d\hat{H}^s_t(u)[\hat{w}]dtds > 0,$$

thus obtaining a contradiction immediately.

Exercise 9.13. Convince yourselves that I am not lying when I say Theorem 9.11 is much easier by supplying a full proof.

Whilst transversality is easier, conversely compactness is slightly more subtle in the s-dependent case than it was in the s-independent case. Let us first examine how the energy behaves in this situation. As before we denote by $\mathcal{N}_\chi^\sharp$ the space of *all* solutions of $\bar{\partial}_\chi(u) = 0$.

$$\mathcal{N}_\chi^\sharp := \left\{ u \in C^\infty(\mathbb{R} \times S^1, Q) \mid u(s, \cdot) \in \mathcal{L}Q,\ \forall\, s \in \mathbb{R},\ \bar{\partial}_\chi(u) = 0 \right\}. \tag{9.4}$$

Definition 9.14. This time we define the *energy* $\mathbb{E} : \mathcal{N}_\chi^\sharp \to [0, +\infty]$ by:

$$\mathbb{E}(u) := \int_{-\infty}^{+\infty} \langle\langle \partial_s u, \partial_s u \rangle\rangle ds.$$

Suppose that $T > 0$ is such that (9.1) holds. Then this time we have

$$\begin{aligned}
\mathbb{E}(u) &= -\int_{-\infty}^{+\infty} d\mathbb{A}_{H^s}(u(s,\cdot)[\partial_s u(s,\cdot)]ds \\
&= -\int_{-\infty}^{+\infty} \frac{d}{ds}\mathbb{A}_{H^s}(u(s,\cdot))ds - \int_{-\infty}^{+\infty}\int_{S^1} \frac{\partial H_t^s}{\partial s}(u(s,t))dtds \\
&= \int_{-\infty}^{+\infty} \langle\langle \partial_s u, \partial_s u\rangle\rangle ds - \int_{-T}^{T}\int_{S^1} \frac{\partial H_t^s}{\partial s}(u(s,t))dtds \\
&\leq \mathbb{A}_{H^-}(x^-) - \mathbb{A}_{H^+}(x^+) + \Delta(H^s)
\end{aligned}$$

where

$$\Delta(H^s) = \sup_{s\in[-T,T]} \sup_{t\in S^1} \sup_{q\in Q} \left| \frac{\partial H_t^s}{\partial s}(q) \right|.$$

As before we denote by $\mathcal{N}_\chi$ the subset of $\mathcal{N}_\chi^\sharp$ with finite energy, and then we have the following analogue of Theorem 8.8.

Theorem 9.15. *The space $\mathcal{N}_\chi$ is compact in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times S^1, Q)$.*

The reader should have no difficulty in checking that the proof of Theorem 8.8 goes through without change:

Exercise 9.16. Prove Theorem 9.15.

Let us now state and prove the analogue of Theorem 8.23.

Theorem 9.17. *Suppose $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose χ is a regular asymptotically constant path connecting (H^-, J^-) to (H^+, J^+). Fix $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$ and suppose that*

$$\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+) = m.$$

Suppose $(u_k) \subset \mathcal{N}_\chi(x^-, x^+)$ has no convergent subsequence. Then necessarily $m \geq 1$, and there exist:

1. *integers $q, p \in \mathbb{N}\cup 0$ such that $1 \leq q + p \leq m$,*
2. *if $q \geq 1$:*
    - *elements $x^- = y^0, \dots, y^q$ of $\mathcal{P}_1^\circ(H^-)$,*
    - *sequences s_k^j of real numbers, for $1 \leq j \leq q$ such that $s_k^j \to -\infty$ and such that $s_k^{j+1} - s_k^j \to -\infty$,*
    - *flow lines $u^j \in \mathcal{M}_{(H^-,J^-)}(y^j, y^{j+1})$ for $1 \leq j \leq q$,*
3. *if $p \geq 1$:*
    - *elements $z^0, \dots, z^p = x^+$ of $\mathcal{P}_1^\circ(H^+)$,*
    - *sequences σ_k^j of real numbers, for $1 \leq j \leq p$ such that $\sigma_k^j \to +\infty$ and such that $\sigma_k^{j+1} - \sigma_k^j \to +\infty$,*
    - *flow lines $v^j \in \mathcal{M}_{(H^+,J^+)}(z^j, z^{j+1})$ for $1 \leq j \leq p$,*

such that after passing to a subsequence, one has

$$\lim_{k\to+\infty} u_k(\cdot + s_k^j, \cdot) = u^j \quad \text{in} \quad C^\infty_{\mathrm{loc}}(\mathbb{R}\times S^1, Q), \quad \text{for} \quad 1 \leq j \leq q,$$
$$\lim_{k\to+\infty} u_k(\cdot + \sigma_k^j, \cdot) = v^j \quad \text{in} \quad C^\infty_{\mathrm{loc}}(\mathbb{R}\times S^1, Q), \quad \text{for} \quad 1 \leq j \leq p,$$

We begin with a lemma. As was implicit in the statement of Theorem 9.17, in what follows we denote by $\mathcal{M}_{(H^\pm,J^\pm)}$ the space $\mathcal{M}$ (as defined in Definition 8.6) for the regular pair $(H^\pm, J^\pm)$.

Lemma 9.18. *Suppose $(u_k) \subset \mathcal{N}_\chi(x^-, x^+)$ and $(s_k) \subset \mathbb{R}$ is a sequence such that $s_k \to +\infty$. Then after passing to a subsequence, there exists $v \in \mathcal{M}_{(H^+,J^+)}$ such that $u_k(\cdot + s_k, \cdot)$ converges to v in the C^∞_{loc}-topology. If instead $s_k \to -\infty$ then after passing to a subsequence there exists $v \in \mathcal{M}_{(H^-,J^-)}$ such that $u_k(\cdot + s_k, \cdot)$ converges to v in the C^∞_{loc}-topology.*

Proof. Recall from Exercise 9.16[6] that there exists a constant $C > 0$ such that $|\nabla u_k(s,t)|_{J^s} \leq C$ for all $(s,t) \in \mathbb{R}$ and all $k \in \mathbb{N}$. Thus the sequence (u_k) is equicontinuous, and since Q is compact, we can again apply the Arzela-Ascoli theorem to deduce that the sequence $u_k(\cdot + s_k, \cdot)$ has a subsequence which converges in C^0_{loc} to some map $v \in C^0(\mathbb{R} \times S^1, Q)$. The usual elliptic regularity results imply that v is of class C^∞. It remains to check that $v \in \mathcal{M}_{(H^+,J^+)}$. Since χ is asymptotically constant, there exists $T > 0$ such that $\chi(s) = (H^+, J^+)$ for all $s \geq T$. Fix a compact subset $I \subset \mathbb{R}$. There exists $k_0 > 0$ such that if $k \geq k_0$ and $s \in I$ then $s + s_k > T$. Thus for all $k \geq k_0$ the curve $v_k := u_k(\cdot + s_k, \cdot)$ satisfies

$$\partial_s v_k + J^+(v_k)\partial_t v_k - J^+(v_k)X_{H_t^+}(v_k) = 0, \qquad \text{on} \qquad I \times S^1.$$

It follows that the limit v satisfies $\bar{\partial}_{J^+,H^+}(v) = 0$ on $I \times S^1$. Since I was arbitrary, we see that $v \in \mathcal{M}^\sharp_{(H^+,J^+)}$. Finally since $\mathbb{E}(u_k) < +\infty$ for each k, it easily follows that $\mathbb{E}(v) < +\infty$, and hence $v \in \mathcal{M}_{(H^+,J^+)}$. ∎

Here is an easy exercise.

Exercise 9.19. Suppose $u \in \mathcal{N}_\chi$. Show there exists $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$ such that

$$\lim_{s \to \pm\infty} u(s, \cdot) = x^\pm$$

in $C^\infty(S^1, Q)$. *Hint: Since χ is asymptotically constant, the proof is exactly the same as the proof of Lemma 8.17.*

We can now complete the proof of Theorem 9.17.

Proof of Theorem 9.17. Recall from the proof of Theorem 8.11 that we can choose $\varepsilon > 0$ such that the balls $B(y;\varepsilon)$ for $y \in \mathcal{P}_1^\circ(H^-)$ are all disjoint, where the balls are taken with respect to the metric (8.8). Up to shrinking ε we can also assume that all the balls $B(z;\varepsilon)$ for $z \in \mathcal{P}_1^\circ(H^+)$ are also all disjoint. Since Theorem 9.15 tells us that the space $\mathcal{N}_\chi$ is compact, there exists $w \in \mathcal{N}_\chi$ such that $u_k \to w$. By Exercise 9.19 there exists two orbits $y \in \mathcal{P}_1^\circ(H^-)$ and $z \in \mathcal{P}_1^\circ(H^+)$ such that $w \in \mathcal{N}_\chi(y, z)$. Then there exists $\bar{s} \in \mathbb{R}$ such that $w(s, \cdot) \in B(z;\varepsilon)$ for all $s \geq \bar{s}$. Thus for all k sufficiently large, we have $u_k(\bar{s}, \cdot) \in B(z;\varepsilon)$. If $z \neq x^+$ then since $u_k \in \mathcal{N}_\chi(x^-, x^+)$, there is a well defined finite number

$$\sigma_k := \sup\left\{ s \geq \bar{s} \mid u_k(\sigma, \cdot) \in B(z;\varepsilon) \text{ for all } \sigma \in [\bar{s}, s] \right\}$$

(compare (8.14)). Suppose that σ_k is bounded. Then we may assume that $\sigma_k \to \sigma_0$ for some $\sigma_0 \in \mathbb{R}$. Since the convergence of u_k to w is in C^∞_{loc}, it would then follow that $\lim_{k \to +\infty} u_k(\sigma_k, \cdot) = w(\sigma_0, \cdot)$. Since necessarily $\sigma_0 \geq \bar{s}$, this would imply that

$$\lim_{k \to +\infty} u_k(\sigma_k, \cdot) \in B(z;\varepsilon).$$

[6]Or more realistically, given that you clearly didn't do the exercise, from the proof of Lemma 8.18.

But this contradicts the definition of σ_k: by definition $u_k(\sigma_k, \cdot) \in \partial B(z; \varepsilon)$. Thus $\sigma_k \to +\infty$. Then we can apply Lemma 9.18 to see that there exists $v \in \mathcal{M}_{(H^+, J^+)}$ such that

$$\lim_{k \to +\infty} u_k(\cdot + \sigma_k, \cdot) = v^0.$$

If $s < 0$ then for k large enough we have $\bar{s} < s + \sigma_k < \sigma_k$ and hence $u_k(s + \sigma_k, \cdot) \in B(z; \varepsilon)$. Thus $v^0(s, \cdot) \in B(z; \varepsilon)$ for all $s < 0$. By Theorem 8.11, we have that $v^0 \in \mathcal{M}_{(H^+, J^+)}(z, z^1)$ for some critical point $z^1 \in \mathcal{P}_1^\circ(H^+)$. Now arguing exactly as in the proof of Theorem 8.23 we see that we can continue this finitely many times, finding new flow lines $v^j \in \mathcal{M}_{(H^+, J^+)}(z^j, z^{j+1})$ and sequences σ_k^j such that $u_k(\cdot + \sigma_k^j, \cdot)$ converges to v^j, until eventually we obtain $z^p = x^+$ for some p. Next we examine the other limit. Since $w \in \mathcal{N}_\chi(y, z)$, there exists $\underline{s} \in \mathbb{R}$ such that $w(s, \cdot) \in B(y; \varepsilon$ for all $s \le \underline{s}$. Thus for all k sufficiently large we have $u_k(\underline{s}, \cdot) \in B(y; \varepsilon)$. If $y \ne x^-$ then since $u_k \in \mathcal{N}_\chi(x^-, x^+)$, there is a well defined finite number

$$s_k := \inf \{ s \le \underline{s} \mid u_k(s, \cdot) \in B(y; \varepsilon) \}$$

Then as above one shows that $s_k \to -\infty$, and then applying Lemma 9.18 again tells us that $u_k(\cdot + s_k, \cdot)$ converges to some $u \in \mathcal{M}_{(H^-, J^-)}$. Then as above we see that $u \in \mathcal{M}_{(H^-, J^-)}(y', y)$ for some orbit y', and finally the argument from Theorem 8.23 supplies the required broken flow line whose final limit $y^0 = x^-$. This completes the proof. ∎

Now let us state the corresponding gluing result.

Theorem 9.20 (*Gluing*)**.** *Suppose $x^- \in \mathcal{P}_1^\circ(H^-)$ and x^0, x^+ are elements of $\mathcal{P}_1^\circ(H^+)$ such that*

$$\mathrm{CZ}(x^-) = \mathrm{CZ}(x^0) = \mathrm{CZ}(x^+) + 1.$$

Suppose $u \in \mathcal{N}_\chi(x^-, x^0)$ and $\underline{v} \in \underline{\mathcal{M}}_{(H^+, J^+)}(x^0, x^+)$. Then there exists $\rho_0 > 0$ and a differentiable an embedding

$$\psi : [\rho_0, +\infty) \to \mathcal{N}_\chi(x^-, x^+)$$

such that:

1. *$\psi(\rho)$ converges (in the sense described by Theorem 9.17) to the broken flow line (u, v) as $\rho \to +\infty$, where $v \in \underline{v}$.*

2. *if $w_k \in \mathcal{N}_\chi(x^-, x^+)$ converges to a pair (u, v) as $k \to +\infty$ for some $v \in \underline{v}$ then for all k sufficiently large there exists ρ_k such that $w_k = \psi(\rho_k)$.*

Following in the tradition of Exercise 8.25, we leave the proof of Theorem 9.20 as an exercise for the masochistic reader:

Exercise 9.21. Prove Theorem 9.20. Then do it again, this time with your eyes closed.

We are now in a position to define the map $\Phi_\chi : \mathrm{CF}_*(H^-) \to \mathrm{CF}_*(H^+)$, and to prove that it is a chain map, that is,

$$\Phi_\chi \circ \partial_{J^-} = \partial_{J^+} \circ \Phi_\chi.$$

The reader should compare this to Theorem 1.28 in Morse theory.

Theorem 9.22. *Suppose $x^- \in \mathcal{P}_1^\circ(H^-)$ and $x^+ \in \mathcal{P}_1^\circ(H^+)$ with $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+) + 1$. Then the boundary $\partial \overline{\mathcal{N}_\chi(x^-, x^+)}$ of the compactification $\overline{\mathcal{N}_\chi(x^-, x^+)}$ of the one-dimensional manifold $\mathcal{N}_\chi(x^-, x^+)$ can be identified as:*

$$\partial \overline{\mathcal{N}_\chi(x^-, x^+)} = \left(\bigcup_{y^- \in \mathcal{P}_1^\circ(H^-),\, \mathrm{CZ}(y^-) = \mathrm{CZ}(x^-) - 1} \underline{\mathcal{M}}_{(H^-, J^-)}(x^-, y^-) \times \mathcal{N}_\chi(y^-, x^+) \right)$$
$$\cup \left(\bigcup_{y^+ \in \mathcal{P}_1^\circ(H^+),\, \mathrm{CZ}(y^+) = \mathrm{CZ}(x^-)} \mathcal{N}_\chi(x^-, y^+) \times \underline{\mathcal{M}}_{(H^+, J^+)}(y^+, x^+) \right).$$

Exercise 9.23. Use Theorem 9.17 and Theorem 9.20 to prove Theorem 9.22.

We now define for $x^- \in \mathcal{P}_1^\circ(H^-)$ and $x^+ \in \mathcal{P}_1^\circ(H^+)$ satisfying $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+)$ the number

$$n_\chi(x^-, x^+) := \#_2 \mathcal{N}_\chi(x^-, x^+),$$

and then define

$$\Phi_\chi : \mathrm{CF}_k(H^-) \to \mathrm{CF}_k(H^+)$$

by setting

$$\Phi_\chi \langle x^- \rangle := \sum_{x^+ \in \mathcal{P}_1^\circ(H^+),\, \mathrm{CZ}(x^+) = k} n_\chi(x^-, x^+) \langle x^+ \rangle, \tag{9.5}$$

and then extending by linearity.

Exercise 9.24. Use Theorem 9.22 to prove that the map Φ_χ defined in (9.5) really is a chain map.

Lemma 9.25. *Suppose $(H^-, J^-) = (H^+, J^+) = (H, J)$ and $\chi(s) \equiv (H, J)$ for all $s \in \mathbb{R}$. Then the chain map Φ_χ is the identity (on the chain level).*

Proof. In this case one has $\mathcal{N}_\chi(x^- x^+) = \mathcal{M}_{(H,J)}(x^-, x^+)$. By Corollary 8.3, if $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+)$ then the moduli space $\mathcal{M}_{(H,J)}(x^-, x^+)$ can be non-empty only when $x^- = x^+$, in which case it contains precisely one element: the constant solution $u(s,t) \equiv x^-(t)$. ∎

Exercise 9.26. Use a similar argument to prove the claim made in Remark 9.6: if $\chi(s) = (H, J^s)$ then the map Φ_χ is upper triangular with 1's on the diagonal, that is, (9.3) holds.

To complete the proof of Theorem 9.3 we need to prove that the induced map $\phi_\chi : \mathrm{HF}_*(H^-, J^-) \to \mathrm{HF}_*(H^+, J^+)$ is independent of the asymptotically constant path χ and satisfies the functoriality property (9.2). This requires us to build a *homotopy of homotopies*, as we now describe.

Suppose that $\mathcal{X} := \{\chi_r\}_{0 \le r \le 1}$ is a path of asymptotically constant paths whose asymptotes are independent of r, say from (H^-, J^-) to (H^+, J^+). Thus we have two chain maps

$$\Phi_{\chi_0} : \mathrm{CF}_*(H^-) \to \mathrm{CF}_*(H^+),$$
$$\Phi_{\chi_1} : \mathrm{CF}_*(H^-) \to \mathrm{CF}_*(H^+),$$

We would like to prove that these two maps are *chain homotopic*, that is, there exists a map

$$P_{\mathcal{X}} : \mathrm{CF}_*(H^-) \to \mathrm{CF}_{*+1}(H^+) \tag{9.6}$$

such that

$$\Phi_{\chi_1} - \Phi_{\chi_0} = P_{\mathcal{X}} \circ \partial_{J^-} + \partial_{J^+} \circ P_{\mathcal{X}}. \tag{9.7}$$

Let us write $\chi_r(s) = (H^{r,s}, J^{r,s})$. Assume $T > 0$ is such that $\chi_r(s)$ is independent of both r and s (and equal to $(H^\pm, J^\pm)$) for $|s| \ge T$.

Definition 9.27. Fix orbits $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$ and consider the space $\mathcal{O}_\mathcal{X}(x^-, x^+)$ of pairs

$$\mathcal{O}_\mathcal{X}(x^-, x^+) := \left\{ (u, r) \mid u \in \mathcal{N}_{\chi_r}(x^-, x^+) \right\}. \tag{9.8}$$

Thus if $\pi : \mathcal{O}_\mathcal{X}(x^-, x^+) \to [0, 1]$ denotes the projection $\pi(u, r) = r$ then $\pi^{-1}(r)$ is the moduli space $\mathcal{N}_{\chi_r}(x^-, x^+)$.

Our aim is to show that $\mathcal{O}_\mathcal{X}(x^-, x^+)$ is a cobordism between the manifolds $\mathcal{N}_{\chi_0}(x^-, x^+)$ and $\mathcal{N}_{\chi_1}(x^-, x^+)$. For this, the first step is to show that $\mathcal{O}_\mathcal{X}(x^-, x^+)$ is indeed a manifold of dimension one more than that of $\mathcal{N}_{\chi_0}(x^-, x^+)$ and $\mathcal{N}_{\chi_1}(x^-, x^+)$ (so that the notion of cobordism makes sense). The proof of this assertion is similar to the proof of Theorem 9.8 and Theorem 9.11, and we content ourselves here with stating the result.

Theorem 9.28. *Suppose that $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose χ_0 and χ_1 are two asymptotically constant regular paths connecting (H^-, J^-) to (H^+, J^+). Suppose moreover that $\mathcal{X} = \{\chi_r\}_{0 \le r \le 1}$ is a path of asmyptotically constant paths connecting χ_0 to χ_1. Write $\chi_r = (H^{r,s}, J^{r,s})$. Then for any $\varepsilon > 0$ there exists a new path $\tilde{\mathcal{X}} = \{\tilde{\chi}_r\}$ such that:*

1. *$\tilde{\chi}_0 = \chi_0$ and $\tilde{\chi}_1 = \chi_1$,*
2. *$\tilde{\chi}_r$ is of the form $(\tilde{H}^{r,s}, J^{r,s})$ (i.e. we are only perturbing the Hamiltonian),*
3. *$\|H^{r,s} - \tilde{H}^{r,s}\|_{C^\infty(S^1 \times Q)} < \varepsilon$ for all $(r, s) \in [0, 1] \times \mathbb{R}$.*
4. *For every pair of orbits $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$, the spaces $\mathcal{O}_\mathcal{X}(x^-, x^+)$ are manifolds with boundary of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+) + 1$, whose boundary is precisely*

$$\partial \mathcal{O}_\mathcal{X}(x^-, x^+) = \left(\mathcal{N}_{\chi_0}(x^-, x^+) \times \{0\} \right) \cup \left(\mathcal{N}_{\chi_1}(x^-, x^+) \times \{1\} \right). \tag{9.9}$$

Exercise 9.29. Prove Theorem 9.28. *Hint: See for instance [AD10, Section 11.3.b] if you get stuck. However you should not get stuck, since this is easy. By now we have proved so many similar results I'm sure you can all do this in your sleep. It should be a bit like rereading a familiar old book, or perhaps like dating an old ex again. Everything should be comfortably familiar, with no nasty surprises.*

Unsurprisingly, we say that a path $\mathcal{X}$ which satisfies the conclusions of Theorem 9.28 is a *regular* path of asymptotically constant paths. Next we discuss compactness. Let us denote by $\mathcal{O}_\mathcal{X}$ the space of all finite energy flow lines:

$$\mathcal{O}_\mathcal{X} = \bigcup_{r \in [0,1]} (\mathcal{N}_{\chi_r} \times \{r\}) = \bigcup_{x^\pm \in \mathcal{P}_1^\circ(H^\pm)} \mathcal{N}_{\chi_r}(x^-, x^+).$$

Since the interval $[0, 1]$ is compact, the proof of the next result is no harder than that of Theorem 9.15.

Theorem 9.30. *There exists a constant $C > 0$ such that $\mathbb{E}(u) \le C$ for all $(u, r) \in \mathcal{O}_\mathcal{X}$. The space $\mathcal{O}_\mathcal{X}$ is compact in the C^∞_{loc}-topology.*

Similarly we have the following analogue of Lemma 9.18.

Lemma 9.31. *Fix $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$ and suppose we are given a sequence $(u_k, r_k) \subset \mathcal{O}_\mathcal{X}$. Let $(s_k) \subset \mathbb{R}$ denote a sequence of real numbers such that $s_k \to +\infty$. Then, after possibly passing to a subsequence, there exists $(v, r^*) \in \mathcal{N}_{\chi_1} \times [0, 1]$ such that $r_k \to r^*$ and such that $u_k(\cdot + s_k, \cdot)$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1, Q)$ to v. Similarly if $\sigma_k \to -\infty$ then (up to a subsequence) there exists $(w, r_*) \in \mathcal{N}_{\chi_0} \times [0, 1]$ such that $r_k \to r_*$ and such that $u_k(\cdot + \sigma_k, \cdot)$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1, Q)$ to w.*

Exercise 9.32. Prove Lemma (9.31), by arguing in the same way as the proof of Lemma 9.18, and using the fact that $[0,1]$ is compact.

Next we present the analogue of Theorem 9.17.

Theorem 9.33. *Suppose $(H^\pm, J^\pm) \in \mathcal{HJ}_{\mathrm{reg}}$ are two regular pairs, and suppose $\mathcal{X} = \{\chi_r\}_{0\le r\le 1}$ is a regular path of asymptotically constant paths connecting (H^-, J^-) to (H^+, J^+). Fix $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$ and suppose that*

$$\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+) + 1 = m.$$

Suppose $(u_k, r_k) \subset \mathcal{O}_{\mathcal{X}}(x^-, x^+)$ has no convergent subsequence. Then necessarily $m \ge 1$, and there exist:

1. *integers $q, p \in \mathbb{N} \cup 0$ such that $1 \le q + p \le m$,*
2. *if $q \ge 1$:*
   - *elements $x^- = y^0, \dots, y^q$ of $\mathcal{P}_1^\circ(H^-)$,*
   - *sequences s_k^j of real numbers, for $1 \le j \le q$ such that $s_k^j \to -\infty$ and such that $s_k^{j+1} - s_k^j \to -\infty$,*
   - *flow lines $u^j \in \mathcal{M}_{(H^-,J^-)}(y^j, y^{j+1})$ for $1 \le j \le q$,*
3. *if $p \ge 1$:*
   - *elements $z^0, \dots, z^p = x^+$ of $\mathcal{P}_1^\circ(H^+)$,*
   - *sequences σ_k^j of real numbers, for $1 \le j \le p$ such that $\sigma_k^j \to +\infty$ and such that $\sigma_k^{j+1} - \sigma_k^j \to +\infty$,*
   - *flow lines $v^j \in \mathcal{M}_{(H^+,J^+)}(z^j, z^{j+1})$ for $1 \le j \le p$,*
4. *an element $(w, r_*) \in \mathcal{O}_{\mathcal{X}}(y^q, z^0)$,*

such that after passing to a subsequence, one has

$$\lim_{k\to+\infty} u_k(\cdot + s_k^j, \cdot) = u^j \quad \text{in} \quad C^\infty_{\mathrm{loc}}(\mathbb{R}\times S^1, Q), \quad \text{for} \quad 1 \le j \le q,$$
$$\lim_{k\to+\infty} u_k(\cdot + \sigma_k^j, \cdot) = v^j \quad \text{in} \quad C^\infty_{\mathrm{loc}}(\mathbb{R}\times S^1, Q), \quad \text{for} \quad 1 \le j \le p,$$

and finally such that

$$\lim_{k\to+\infty} (u_k, r_k) = (w, r_*).$$

Exercise 9.34. Prove Theorem 9.33, and then (you know you want to) state and prove a suitable gluing theorem.

The previous results imply the following statement:

Theorem 9.35. *Suppose $x^- \in \mathcal{P}_1^\circ(H^-)$ and $x^+ \in \mathcal{P}_1^\circ(H^+)$. Suppose $\mathcal{X} = \{\chi_r\}_{0\le r\le 1}$ is a regular path of asymptotically constant paths. Then:*

1. *If $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+) - 1$ then the space $\mathcal{O}_{\mathcal{X}}(x^-, x^+)$ is a compact manifold of dimension zero. Thus we can define its parity*

$$n_{\mathcal{X}}(x^-, x^+) := \#_2 \mathcal{O}_{\mathcal{X}}(x^-, x^+). \tag{9.10}$$

2. *If instead one has* $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+)$ *then the boundary* $\partial\overline{\mathcal{O}_{\mathcal{X}}(x^-, x^+)}$ *of the compactification* $\overline{\mathcal{O}_{\mathcal{X}}(x^-, x^+)}$ *of the one-dimensional manifold* $\mathcal{O}_{\mathcal{X}}(x^-, x^+)$ *can be identified as:*

$$\begin{aligned}\partial\overline{\mathcal{O}_{\mathcal{X}}(x^-, x^+)} = \big(\mathcal{N}_{\chi^0}(x^-, x^+) \times \{0\}\big) \cup \big(\mathcal{N}_{\chi^1}(x^-, x^+) \times \{1\}\big) \\ \cup \left(\bigcup_{y^- \in \mathcal{P}_1^\circ(H^-),\, \mathrm{CZ}(y^-) = \mathrm{CZ}(x^-) - 1} \underline{\mathcal{M}}_{(H^-, J^-)}(x^-, y^-) \times \mathcal{O}_{\mathcal{X}}(y^-, x^+) \right) \\ \cup \left(\bigcup_{y^+ \in \mathcal{P}_1^\circ(H^+),\, \mathrm{CZ}(y^+) = \mathrm{CZ}(x^+) + 1} \mathcal{O}_{\mathcal{X}}(x^-, y^+) \times \underline{\mathcal{M}}_{(H^+, J^+)}(y^+, x^+) \right).\end{aligned}$$

Exercise 9.36. Use Theorem 9.33 and the gluing theorem you formulated in Exercise 9.34 to prove Theorem 9.35.

We can now define the so-called *prism operator* from (9.6). Namely, we set

$$P_{\mathcal{X}} \langle x^- \rangle := \sum_{x^+ \in \mathcal{P}_1^\circ(H^+),\, \mathrm{CZ}(x^+) = k+1} n_{\mathcal{X}}(x^-, x^+) \langle x^+ \rangle, \tag{9.11}$$

where the numbers $n_{\mathcal{X}}(x^-, x^+)$ were defined in (9.10), and then extending by linearity.

Exercise 9.37. Use Theorem 9.35 to prove that the map $P_{\mathcal{X}}$ defined in (9.11) satisfies (9.7).

We have now nearly completed the proof of Theorem 9.3: it follows from (9.7) that if χ is a regular asymptotically constant path connecting (H^-, J^-) to (H^+, J^+) then the induced map $\phi_\chi : \mathrm{HF}_*(H^-) \to \mathrm{HF}_*(H^+)$ depends only on the asymptotes $(H^\pm, J^\pm)$. It remains to verify the functoriality property. This states that if (H^0, J^0) is another regular pair, and χ^0 and χ^1 are regular asymptotically constant paths connecting (H^-, J^-) to (H^0, J^0) and (H^0, J^0) to (H^+, J^+) respectively, then the induced maps satisfy

$$\phi_\chi = \phi_{\chi^0} \circ \phi_{\chi^1}. \tag{9.12}$$

This does *not* follow from what we have already proved, but it is only a short step away. Indeed, given such a pair χ^0 and χ^1, write $\chi^0(s) = (H^{0,s}, J^{0,s})$ and similarly $\chi^1(s) = (H^{1,s}, J^{1,s})$, and suppose that both χ^0 and χ^1 are independent of s for $|s| \geq T$. Then for $\tau > T$ we define

$$\chi^\tau(s) := \begin{cases} (H^{0,s+\tau}, J^{0,s+\tau}), & s \leq 0, \\ (H^{1,s-\tau}, J^{1,s-\tau}), & s \geq 0. \end{cases}$$

Note that $\chi^\tau(s)$ is independent of s for $|s| \geq T + \tau$. Thus χ^τ is an asymptotically constant path from χ^0 to χ^1. One can show that, after possibly a small perturbation of χ^0 and χ^1, there exists a sequence $\tau_k \to +\infty$ such that this χ^{τ_k} is a regular asymptotically constant path for each $k \in \mathbb{N}$. See [AD10, Lemma 11.5.1]. Since we already know that the induced map between the Floer homology groups only depends on the asymptotes, we know that our original map ϕ_χ from the left-hand side of 9.12 agrees with $\phi_{\chi^{\tau_k}}$:

$$\phi_\chi = \phi_{\chi^{\tau_k}}, \qquad \text{as maps} \qquad \mathrm{HF}_*(H^-) \to \mathrm{HF}_*(H^+).$$

Thus to complete the proof of (9.12), and hence of Theorem 9.3, it suffices to prove:

Proposition 9.38. *For all k sufficiently large, and for all $x^\pm \in \mathcal{P}_1^\circ(H^\pm)$ such that* $\mathrm{CZ}(x^-) = \mathrm{CZ}(x^+)$*, there is a bijection between the zero-dimensional spaces:*

$$\bigcup_{x^0 \in \mathcal{P}_1^\circ(H^0),\, \mathrm{CZ}(x^0)=\mathrm{CZ}(x^-)} \mathcal{N}_{\chi^0}(x^-, x^0) \times \mathcal{N}_{\chi^1}(x^0, x^+) \qquad \text{and} \qquad \mathcal{N}_{\chi^{\tau_k}}(x^-, x^+).$$

Exercise 9.39. Use Proposition 9.38 to show that for all k sufficiently large, the chain maps $\Phi_{\chi^1} \circ \Phi_{\chi^0}$ and $\Phi_{\chi^{\tau_k}}$ (as defined in (9.5)) coincide.

Remark 9.40. Thus Proposition 9.38 and Exercise 9.39 show that for the "special" path χ^{τ_k}, the maps $\Phi_{\chi^1} \circ \Phi_{\chi^0}$ and $\Phi_{\chi^{\tau_k}}$ coincide even on the chain level. In general however we can only expect 9.12 to hold on the level of homology.

The proof of Proposition 9.38 is a gluing argument, and thus sadly we will omit this. The interested reader is referred to [AD10, Section 11.5] for a complete proof. We have now completed the proof of Theorem 9.3, and hence also of Corollary 9.4. We therefore know that the Floer homology groups $\mathrm{HF}_*(H)$ are canonically independent of the choice of non-degenerate H. Thus all that remains to prove the main result, Theorem 2.25 is to actually sit down and *compute* these groups.

Although we shall not persue this method, the original argument (which is due to Salamon and Zehnder [SZ92]) used to compute the Floer homology groups $\mathrm{HF}_*(H)$ was to show that if $H_t(q) = h(q)$ is an *autonomous* Hamiltonian on Q which is a Morse function satisfying the C^2-smallness assumption:

$$\|dX_h\|_{L^2} < 2\pi \tag{9.13}$$

then the Floer homology groups $\mathrm{HF}_*(h)$ agree with the Morse homology groups $\mathrm{HM}_{*+n}(h) \cong \mathrm{H}_{*+n}(Q; \mathbb{Z}_2)$. To see why this might be reasonable, we observe that (9.13) implies that h is non-degenerate: in fact in this case every element x of $\mathcal{P}_1^\circ(h)$ is constant, $x(t) \equiv q$ for some $q \in \mathrm{crit}(h)$. Thus in particular h is non-degenerate (as h is assumed to be Morse).

To prove this we may assume that h is a function on $\mathbb{R}^{2n}$. If $x(t)$ is a 1-periodic solution, we can represent x by its Fourier expansion:

$$x(t) = \sum_{k \in \mathbb{Z}} a_k(x) e^{2\pi i k t}.$$

Thus

$$\dot{x}(t) = \sum_{k \in \mathbb{Z}} 2\pi i k a_k(x) e^{2\pi i k t}.$$

Using Parseval's inequality we obtain

$$\|\ddot{x}\|^2_{L^2(S^1)} = \sum_{k \in \mathbb{Z}} 4\pi^2 k^2 |a_k(\dot{x})|^2 \geq 4\pi^2 \sum_{k \neq 0} |a_k(\dot{x}|^2 = 4\pi^2 \|\dot{x}\|^2_{L^2(S^1)},$$

since necessarily $a_k(\dot{x}) = 0$. Thus we obtain

$$\|\dot{x}\|_{L^2(S^1)} \leq \frac{1}{2\pi} \|\ddot{x}\|_{L^2(S^1)}.$$

But the assumption (9.13) implies that if $x \in \mathcal{P}_1(h)$ is non-constant, then since $\ddot{x} = dX_h(x)(\dot{x})$, one has

$$\|\ddot{x}\|_{L^2(S^1)} < 2\pi \|\dot{x}\|_{L^2(S^1)}.$$

Comparing the last two equations we see that every $x \in \mathcal{P}_1(h)$ is necessarily constant.

Suppose now that $u : \mathbb{R} \times S^1 \to Q$ solves the Floer equation

$$\partial_s u + J(u)\partial_t u - J(u) X_h(u) = 0.$$

If we could show that $\partial_t u \equiv 0$, then since $JX_h = \nabla h$ (using the metric $g_J := \omega(J\cdot, \cdot)$), it would follow that u is a *positive* gradient flow line of ∇h (and hence a negative gradient flow line of $\nabla(-h)$). This is not such an unreasonable hope, since we have just shown that all critical points of the Floer equation are independent of t under the assumption (9.13), and hence it is not too much to hope that the same is true for all flow lines too. This would immediately imply that Floer complex agrees with the Morse complex for $(-h, g_J)$ (since we have positive gradient flow lines). Modulo identifying the grading shift, this would show that the Floer homology agrees with the Morse homology, and hence the singular homology.
Unfortunately showing that all the gradient flow lines are independent of t is very difficult, and we refer the reader to [AD10, Chapter 10] for a complete exposition. In particular we emphasise that whilse (9.13) is enough to ensure that all critical points are constant, in order to prove all flow lines are independent of t one needs to strengthen (9.13). Namely, one can prove that there exists $2\pi \gg \varepsilon > 0$ such that if $\|dX)h\|_{L^2} < \varepsilon$ then all gradient flow lines are independent of t.
Instead we will argue somewhat differently, by making use of the Morse-Bott theory developed at the end of Section 1. Let us begin by defining the notion of a *weakly non-degenerate*[7] Hamiltonian.

Definition 9.41. Fix $H \in C^\infty(S^1 \times Q)$, but do not assume that H is non-degenerate. Let us write $\lambda : \mathcal{P}_1^\circ(H) \to Q$ for the map $\lambda(x) := x(0)$. A connected component $K \subset \mathcal{P}_1^\circ(H)$ is called a *Morse-Bott* component if $\lambda(K)$ is a closed submanifold of Q such that

$$T_q\lambda(K) = \ker(D\varphi_H(q) - \mathbb{1}) \qquad \text{for all} \qquad q \in \lambda(K). \tag{9.14}$$

Thus if $K = \{x\}$ consists of a single loop then K is a Morse-Bott component if and only if x is a non-degenerate element of $\mathcal{P}_1^\circ(H)$ in the sense of Definition 2.13. A Hamiltonian $H \in C^\infty(S^1 \times Q)$ is called *weakly non-degenerate* if every component of $\mathcal{P}_1^\circ(H)$ is a Morse-Bott component.

Remark 9.42. Note that we are *not* insisting that every component $K \subset \mathcal{P}_1^\circ(H)$ is a manifold of the same dimension.

The Morse-Bott condition can be reformulated as follows. First, a subset $K \subset \mathcal{P}_1^\circ(H)$ is a compact submanifold of the loop space ΛQ if and only if the set $\lambda(K)$ is a compact submanifold of Q. Now recall from Lemma 3.15 that the Hessian $D^v\nabla_J\mathbb{A}_H(x)$ of the Hamiltonian action functional $\mathbb{A}_H$ at a critical point $x \in \mathcal{P}_1^\circ(H)$ is given by

$$D^v\nabla_J\mathbb{A}_H(x)[\xi] := J(x)(\nabla_t\xi - \nabla_\xi X_{H_t}(x)).$$

Next, recall from Lemma 3.16 that for every $x \in K$ the kernel of the linear map $D\varphi_H(x(0))$ on $T_{x(0)}Q$ is isomorphic to the kernel of the Hessian of $\mathbb{A}_H$ at x: namely if $D^v\nabla_J\mathbb{A}_H(x)[\xi] = 0$ then $\xi(t) = D\varphi_H^t(x(0)[\xi(0)]$. Thus the Morse-Bott condition asserts that the kernel of the Hessian agrees with the tangent space of the critical manifold K. This proves:

Lemma 9.43. *A Hamiltonian $H \in C^\infty(S^1 \times Q)$ is weakly non-degenerate if and only if the action functional $\mathbb{A}_H$ is a Morse-Bott function on ΛQ in the sense of Definition 1.42.*

We will now explain how to construct the Morse-Bott Floer complex. If H is weakly non-degenerate then the space $\mathrm{crit}(\mathbb{A}_H)$ is a closed finite-dimensional submanifold of ΛQ. Let us fix a Morse function

$$h : \mathrm{crit}(\mathbb{A}_H) \to \mathbb{R}.$$

[7] *Warning:* This is *not* standard terminology. I just made it up!

As in Section 1, what we really mean by this is that for each component $K \subset \mathcal{P}_1^\circ(H)$, we select a Morse function $h_K : K \to \mathbb{R}$, and then we denote by $h : \mathrm{crit}(\mathbb{A}_H) \to \mathbb{R}$ the function defined by $h|_K := h_K$. We then choose a Riemannian metric ρ on $\mathrm{crit}(\mathbb{A}_H)$, such that the negative gradient flow ϕ^s of h with respect to ρ is Morse-Smale. We denote by $W^u(x; -\nabla_\rho h)$ the *unstable manifold* of x with respect to the flow ϕ^s:

$$W^u(x; -\nabla_\rho h) := \left\{ y \in \mathrm{crit}(\mathbb{A}_H) \mid \lim_{s \to -\infty} \phi^s(y) = x \right\}.$$

Similarly the *stable manifold* $W^s(x; -\nabla_\rho h)$ is the set

$$W^s(x; -\nabla_\rho h) := \left\{ y \in \mathrm{crit}(\mathbb{A}_H) \mid \lim_{s \to +\infty} \phi^s(y) = x \right\}.$$

Definition 9.44. Fix critical points $x^\pm \in \mathcal{P}_1^\circ(H)$. An element of $\mathcal{M}_k^c(x^-, x^+)$ is a tuple $(2k-1)$-tuple

$$(\mathbf{u} = (u_1, \dots, u_k), \mathbf{t} = (t_1, \dots, t_{k-1})),$$

where $u_j \in C^\infty(\mathbb{R} \times S^1, Q)$ and $t_j \geq 0$ are such that:

1. Each u_j is a non-constant gradient flow line of $\mathbb{A}_H$:
$$\partial_s u_j + J(u)(\partial_t u - X_{H_t}(u_j)) = 0.$$
2. The first flow line u_1 satisfies
$$\lim_{s \to -\infty} u_1(s) \in W^u(x; -\nabla_\rho h),$$
and the last flow line u_k satisfies
$$\lim_{s \to +\infty} u_k(s) \in W^s(x^+; -\nabla_\rho h).$$
3. For $1 \leq j \leq k-1$ there are critical submanifolds K_{i_j} and gradient flow lines $v_j \in C^\infty(\mathbb{R}, K_{i_j})$ of h:
$$\partial_s v_j + \nabla_\rho h(v_j) = 0,$$
such that
$$\lim_{s \to +\infty} u_j(s) = v_j(0),$$
$$\lim_{s \to -\infty} u_{j+1}(s) = v_j(t_j).$$

The 'c' in $\mathcal{M}^c$ stands for 'cascades'. There is a free $\mathbb{R}$-action on each flow line u_j, and hence $\mathcal{M}_k(x^-, x^+)$ admits a free $\mathbb{R}^k$-action. As before we denote by $\underline{\mathcal{M}}_k^c(x^-, x^+)$ the quotient space

$$\underline{\mathcal{M}}_k^c(x^-, x^+) := \mathcal{M}_k^c(x^-, x^+)/\mathbb{R}^k.$$

If $l = m$ then we let $\mathcal{M}_0(x^-, x^+)$ denote the set of normal gradient flow lines of h running from x^- to x^+, and as usual $\underline{\mathcal{M}}_0^c(x^-, x^+)$ is then the quotient space $\mathcal{M}_0^c(x^-, x^+)/\mathbb{R}$. If x^- and x^+ do not belong to the same component then we set $\mathcal{M}_0(x^-, x^+) := \emptyset$. Finally we set

$$\mathcal{M}^c(x^-, x^+) := \bigcup_{k=0}^{\infty} \mathcal{M}_k^c(x^-, x^+),$$

and

$$\underline{\mathcal{M}}^c(x^-,x^+) := \bigcup_{k=0}^{\infty} \underline{\mathcal{M}}^c_k(x^-,x^+)$$

In other words, the space $\underline{\mathcal{M}}^c(x^-,x^+)$ is the space of *gradient flow lines with arbitrarily many cascades*. Recall we defined the Conley-Zehnder index $\mathrm{CZ}(x)$ for non-degenerate elements of $\mathcal{P}_1^\circ(H)$. In fact, it is possible to extend the definition of CZ to deal with Morse-Bott components. However in this case $\mathrm{CZ}(x)$ is only a half-integer in general. The precise definition of this extension would take too long to explain properly in these notes however. Thus for now the reader is invited to simply pretend that we have also defined $\mathrm{CZ}(x)$ in this more general case. See [RS93, RS95] for the proper definition.

Definition 9.45. Given a component $K \subset \mathcal{P}_1^\circ(H)$, we define

$$\mu(K) := \mathrm{CZ}(x) - \frac{1}{2}\dim K,$$

where $x \in K$ (one can show that CZ is constant on K, and moreover that $\mu(K)$ is always an integer). Given $x \in \mathrm{crit}(h) \subset \mathcal{P}_1^\circ(H)$ we define

$$\mu_h(x) := \mu(K) + \mathrm{ind}_h(x),$$

where as usual $\mathrm{ind}_h(x)$ denotes the index of x as a critical point of h.

The following result is an extension of the index theory we discussed earlier:

Theorem 9.46. *Fix two components K^- and K^+ of $\mathcal{P}_1^\circ(H)$ and let $\mathcal{M}(K^-,K^+)$ denote the space of all gradient flow lines u of $\mathbb{A}_H$ running from some point $x^- \in K^-$ to some point $x^+ \in K^+$:*

$$\mathcal{M}(K^-,K^+) := \bigcup_{x^\pm \in K^\pm} \mathcal{M}(x^-,x^+)$$

(note we are not dividing out by the $\mathbb{R}$ -action). Then $\mathcal{M}(K^-,K^+)$ has virtual dimension

$$\mathrm{virdim}\,\mathcal{M}(K^-,K^+) = \mu(K^-) - \mu(K^+) + \dim K^-.$$

We won't prove Theorem 9.46. The proof is not particular difficult, but it is not possible to give without first giving the definition of CZ in this more general setting. Having proved Theorem 9.46 one can then deduce the following result.

Theorem 9.47. *The space $\underline{\mathcal{M}}^c(x^-,x^+)$ has virtual dimension*

$$\mathrm{virdim}\,\underline{\mathcal{M}}^c(x^-,x^+) = \mu_h(x^-) - \mu_h(x^+) - 1.$$

Skipping lightly over numerous difficulties, as in Section 1 one can then proceed with the construction of the Morse-Bott Floer groups. First we define the chain complex

$$\mathrm{CFB}_*(H,h) := \bigoplus_{x\in\mathrm{crit}(h)} \mathbb{Z}_2\langle x\rangle,$$

where the grading is given by μ_h, and then one defines the boundary operator

$$\partial_{J,\rho} : \mathrm{CFB}_k(H,h) \to \mathrm{CFB}_*(H,h)$$

by setting

$$\partial\langle x\rangle := \sum_{y\in\mathrm{crit}(h),\, \mu_h(y)=\mu_h(x)-1} n^c(x,y)\langle y\rangle\,,$$

and then extending by linearity. Here $n^c(x,y)$ denotes the parity of the moduli space $\underline{\mathcal{M}}^c(x,y)$. This is well defined, since one can prove that these moduli spaces are compact in dimension zero. By studying the compactness properties of the one-dimensional moduli spaces one proves that $\partial \circ \partial = 0$, and thus we again obtain a homology theory. Moreover the theory of continuation maps developed above also goes through, and so as before we see that this homology is independent of all the auxilliary data (H, h, J, ρ). Moreover since a non-degenerate Hamiltonian is also a weakly non-degenerate Hamiltonian, this new Floer homology also contains the one considered previously as a special case. We summarize this in the following theorem.

Theorem 9.48. *The Floer homology groups* $\mathrm{HF}_*(H)$ *are also well defined if* H *is assumed only to be weakly non-degenerate, and moreover they are independent of the weakly non-degenerate Hamiltonian* H*.*

The reader may question why there was any point in introducing the Morse-Bott theory. The following exercise explains this:

Exercise 9.49. Show that the zero Hamiltonian $H \equiv 0$ is weakly non-degenerate.

In this case one has $\mathcal{P}_1^\circ(H) \cong Q$ consisting exactly of the constant loops. Thus choosing a Morse function $h : \mathcal{P}_1^\circ(H) \to \mathbb{R}$ is the same thing as choosing a Morse function h on Q. For q a constant loop one has $\mathrm{CZ}(q) = 0$, and hence the grading $\mu_h(q)$ is given by $\mathrm{ind}_h(q) - n$. Next, since all critical points are constant, and $\mathbb{A}_H(q) = 0$ for any constant q, we see that there are no non-constant gradient flows lines of $\mathbb{A}_H$ (recall that $\mathbb{A}_H$ strictly decreases along non-constant gradient flow lines). Thus the Morse-Bott Floer complex reduces to the Morse complex of h, and we obtain:

Theorem 9.50. *One has*

$$\mathrm{HF}_*(H \equiv 0, h) \cong \mathrm{HM}_{*+n}(h) \cong \mathrm{H}_{*+n}(Q; \mathbb{Z}_2).$$

Combining the last two theorems we see that we have finally completed the proof of Theorem 2.25. Thus we have also proved the non-degenerate Arnold Conjecture 2.16 for symplectically aspherical symplectic manifolds.

CHAPTER 10

Floer homology of cotangent bundles

10.1 Hamiltonian dynamical systems on cotangent bundles

Let B be a connected closed orientable smooth manifold of dimension n. Points in the cotangent bundle T^*B will be denoted by (q,p), with $q \in B$, $p \in T^*_qB$, and $\pi : T^*B \to B$ will denote the foot point map. Similarly points in the tangent bundle TB are denoted by (q,v), and by a slight abuse of notation we denote also by π the foot point map $TB \to B$. The cotangent bundle T^*B carries the following canonical structures: the *Liouville 1-form* λ and the Liouville vector field Z, which are defined by

$$\lambda(\xi) = p(D\pi(x)[\xi]) = d\lambda(Z,\xi) \quad \forall \xi \in T_xT^*B,\ x=(q,p) \in T^*B,$$

and the symplectic structure $\omega = d\lambda$. In local coordinates $(q_1,\dots,q_n,p_1,\dots p_n)$ of T^*B we have

$$\lambda_{\text{local}} = \sum_{j=1}^n p_j\, dq_j, \quad Z_{\text{local}} = \sum_{j=1}^n p_j \frac{\partial}{\partial p_j}, \quad \omega_{\text{local}} = \sum_{j=1}^n dp_j \wedge dq_j.$$

The vertical space

$$T^v_xT^*B = \ker D\pi^*(x) \cong T^*_qB, \quad x=(q,p)\in T^*B,$$

is a Lagrangian subspace of (T_xT^*B, ω_x).

A 1-periodic Hamiltonian H, i.e. a smooth function $H : S^1 \times T^*B \to \mathbb{R}$, determines as usual a 1-periodic vector field X_H, and in local coordinates the Hamiltonian equations are familiar from classical mechanics:

$$\dot{x}(t) = X_H(t,x(t)), \tag{10.1}$$

writing $x(t) = (q(t),p(t))$ becomes:

$$\begin{cases} \dot{q} = \partial_p H(t,q,p), \\ \dot{p} = -\partial_q H(t,q,p). \end{cases} \tag{10.2}$$

10.2 The geometry of the (co)tangent bundle

Let us now fix an auxilliary Riemannian metric g on B. Let ∇ denote the Levi-Civita connection of g.

Definition 10.1. We define the *connection map* κ_g of g as follows. Fix $\xi \in T_wTB$ and choose a curve $w : (-\varepsilon,\varepsilon) \to TB$, $w(t) = (q(t),v(t))$ adapted to ξ. This means that $w(0) = w$ and $\dot{w}(0) = \xi$. Then define

$$\kappa_g(\xi) := (\nabla_{\dot{q}} v)(0) = (\nabla_t v)(0).$$

The connection map κ_g defines a *horizontal-vertical* splitting of $T(TB)$: given $w = (q,v) \in T(TB)$ we write

$$T_wTB = T^h_wTB \oplus T^v_wTB \cong T_qB \oplus T_qB, \tag{10.3}$$

where $T^h_wTB = \ker(\kappa_g : T_{(q,v)}TB \to T_qB)$, where κ_g is the *connection map* of the Levi-Civita connection ∇ of g.

Given $\xi \in T(TB)$ we denote by ξ^h and ξ^v the horizontal and vertical components, and given $\xi \in T_{(q,v)}TB$ and $u, v \in T_qB$, we will often write

$$\xi \approx (u, v) \tag{10.4}$$

to indicate that under the splitting (10.3) one has $\xi^h = u$ and $\xi^v = v$.

Note that it if $w(t) = (q(t), v(t))$ is a curve in TB then by definition one has

$$\dot{w}^h = \dot{q}, \qquad \dot{w}^v = \nabla_t v, \tag{10.5}$$

and hence $\dot{w} \approx (\dot{q}, \nabla_t v$. The *Sasaki metric* g_{TB} on TB is defined by

$$g_{TB}(\xi, \zeta) := \left\langle \xi^h, \zeta^h \right\rangle + \langle \xi^v, \zeta^v \rangle .$$

Warning: We warn the reader now that we will consistently use the "musical" isomorphism $v \mapsto \langle v, \cdot \rangle$ to identify TB with T^*B. Thus we will frequently refer to the above splitting also as a splitting of $T(T^*B) = T^hT^*B \oplus T^vT^*B$. In particular, we will talk about the Sasaki metric g_{T^*B} on T^*B. Moreover, unless specified otherwise, the norm $|\cdot|$ should always be thought as being with respect to g if the (co)vector belongs to TB or T^*B, and with respect to g_{TB} or g_{T^*B} if the vector belongs to $T(TB)$ or $T(T^*B)$.

The horizontal-vertical splitting also determines an almost complex structure J_g called the *metric almost complex structure* via

$$J_g = \begin{pmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{pmatrix}.$$

The metric almost complex structure J_g is ω-compatible. Indeed, $\omega(J_g\cdot, \cdot)$ is precisely the Sasaki metric g_{T^*B}.

10.3 The functional setting

As we will see, for cotangent bundles there is no need to restrict to just the contractible loops. Thus let us be slightly naughty and redefine the space ΛT^*B as the space $C^\infty(S^1, T^*B)$ of *all* smooth loops, rather than just the contractible ones. Similarly we will denote by ΛB the space of all smooth maps from S^1 into B. Unfortunately as we have already noted the spaces ΛB and ΛT^*B do not admit the structure of a Hilbert manifold (they are only Fréchet manifolds, which is not sufficient for Morse theory), and hence we will often work with their Sobolev completions. Although we have already dealt with these spaces, the treatment now needs to be more thorough, and so we define them properly. For our purposes it is most convenient to define these spaces in the following manner.

By Nash's embedding theorem, we may view (B, g) as being embedded isometrically in some $(\mathbb{R}^N, g_e)$, where g_e denotes the Euclidean metric on $\mathbb{R}^N$.

Let $W^{1,2}([0,1], \mathbb{R}^N)$ denote the Hilbert manifold

$$W^{1,2}([0,1], \mathbb{R}^N) := \left\{ q : [0,1] \to \mathbb{R}^N \text{ absolutely continuous} \mid \int_0^1 |\dot{q}(t)|^2 \, dt < \infty \right\},$$

equipped with the Hilbert product from Definition 3.3.

$$\langle\langle \xi, \zeta \rangle\rangle_{1,2} := \int_0^1 \langle \xi, \zeta \rangle \, dt + \int_0^1 \left\langle \dot{\xi}, \dot{\zeta} \right\rangle dt. \tag{10.6}$$

Now consider the submanifold

$$W^{1,2}([0,1],B) := \left\{ q \in W^{1,2}([0,1],\mathbb{R}^N) \mid q(t) \in B \text{ for almost every } t \in [0,1] \right\}.$$

These spaces are defined independently of the embedding and of the metric g on B. Moreover if $\xi, \zeta \in C^\infty(q^*W)$ for $q \in W^{1,2}([0,1],B)$ then the Hilbert product $\langle\langle \eta, \eta' \rangle\rangle_{1,2}$ coincides with the Hilbert product from Definition 3.3, where we use the Levi-Civita connection of g.

Let

$$\mathcal{L}\mathbb{R}^N := \{q \in W^{1,2}([0,1],\mathbb{R}^N) \mid q \text{ is closed}\}.$$

and let $\mathcal{L}B$ denote the submanifold of $W^{1,2}([0,1],B)$ defined by

$$\mathcal{L}B := W^{1,2}([0,1],B) \cap \mathcal{L}\mathbb{R}^N.$$

We can identify $T_q\mathcal{L}B$ with $W^{1,2}(q^*TB)$, and the space $\mathcal{L}B$ is homotopy equivalent to ΛB . The embedding $(M,g) \hookrightarrow (\mathbb{R}^N, g_e)$ induces an embedding $(T^*B, g_{T^*B}) \hookrightarrow (\mathbb{R}^{2N}, g_e)$. We define $W^{1,2}([0,1],T^*B)$ using this embedding, and from that the spaces $\mathcal{L}T^*B$. The nice thing about these choices of embeddings is that they mean that all norms we work with (i.e. coming from the metric g or the Sasaki metric g_{T^*B}) coincide with the Euclidean norms. Thus in all of what follows, one can happily interpret all norm signs $|\cdot|$ as being the Euclidean norm. In particular, since we are now in a Euclidean setting we can work with Sobolev spaces that are in general not defined, such as $W^{1,2}(\mathbb{R}\times[0,1],T^*B)$. Note however that this latter space is *not* invariantly defined (and thus depends on our specific choice of embedding).

Exercise 10.2. Prove that under this embedding if J_0 denotes the canonical almost complex structure on $\mathbb{R}^{2N}$ from (3.3) then $J_0|_{T^*B} = J_g$.

10.4 Lagrangians and Hamiltonians

Suppose $L \in C^\infty(S^1 \times TB, \mathbb{R})$ is a smooth function. For historical reasons, just as smooth functions on cotangent bundles are called "Hamiltonians", smooth functions on the tangent bundle are known as *"Lagrangians"* (not to be confused with Lagrangian submanifolds!).

As usual we will often write $L_t : TB \to \mathbb{R}$ for the function $L(t,\cdot,\cdot)$. We denote by $\nabla L_t = \nabla_{g_{TB}} L_t$ the gradient of L_t with respect to the Sasaki metric g_{TB}. Thus given a point $w = (q,v) \in TB$, the vector $\nabla L_t(w)$ belongs to T_wTB. As a result we can take its horizontal and vertical components, which we denote by

$$\nabla^h L_t(q,v) := (\nabla L_t(q,v))^h \in T_qB, \qquad \nabla^v L_t(q,v) := (\nabla L_t(q,v))^v \in T_qB. \tag{10.7}$$

But now we can play this game again. For instance, think of $\nabla^h L_t$ as a map $TB \to TB$. Then we can consider its differential

$$D(\nabla^h L_t) : T(TB) \to T(TB).$$

The fun doesn't stop here: we use $D(\nabla^h L_t)$ to define for each $(q,v) \in TB$ a map

$$\nabla^{hh} L_t(q,v) : T_qB \to T_qB,$$

by setting

$$\nabla^{hh} L_t(q,v)[u] := \Big([D(\nabla^h L_t)(q,v)[\xi] \Big)^v,$$

where ξ is the unique vector such that $\xi \approx (u, 0)$ (using the notation from (10.4)). Similarly we define a map $\nabla^{hv} L(q, v) : T_q B \to T_q B$ by setting

$$\nabla^{hv} L_t(q, v)[u] := \Big([D(\nabla^h L_t)(q, v)[\zeta] \Big)^v ,$$

where now ζ is the unique vector such that $\zeta \approx (0, u)$. Finally we define maps $\nabla^{vh} L_t$ and $\nabla^{vv} L_t$ in exactly the same way, starting with $\nabla^v L_t$ instead.

Exercise 10.3. Despite the fact I have cleverly managed to make this look very complicated, in fact it is really easy. Convince yourself of this by showing that $\nabla^{vv} L_t(q, v)$ coincides with the second derivative of the map $v \mapsto L_t(q, v)$ in the vector space $T_q B$. Then answer the following question: does one still have "equality of mixed partial derivatives" in this setting? That is, does it hold that $\nabla^{hv} L_t = \nabla^{vh} L_t$?

In exactly the same way given a Hamiltonian $H \in C^\infty(S^1 \times T^* B, \mathbb{R})$ we can speak of the operators $\nabla^h H_t$, $\nabla^v H_t$, $\nabla^{hh} H_t$, $\nabla^{hv} H_t$, $\nabla^{vh} H_t$ and $\nabla^{vv} H_t$.

Definition 10.4. A Lagrangian $L \in C^\infty(S^1 \times TB, \mathbb{R})$ is called a *Tonelli* Lagrangian if the following three conditions are satisfied:

(**T1**) L_t is *fibrewise strictly convex*: that is, the operator $\nabla^{vv} L_t$ is positive definite. Thus for any $(q, v) \in TB$, one has

$$\nabla^{vv} L_t(q, v)(u) > 0 \qquad \text{for all } 0 \neq u \in T_q B.$$

(**T2**) L_t is *fibrewise superlinear*: that is, for any constant $C > 0$ there exists a finite constant $A(C) > 0$ such that

$$L(t, q, v) \geq C|v| - A(C), \qquad \text{for all } (t, q, v) \in S^1 \times TB.$$

Here $|\cdot|$ denotes some Riemannian metric on B, but due to compactness of B the condition (**T2**) does not depend on the choice of metric (up to changing the finite constant $A(C)$).

(**T3**) The Hamiltonian flow $\varphi_H^t : T^* B \to T^* B$ of the Fenchel dual Hamiltonian H (defined in Definition 10.5 immediately below) exists for all $(t, q, p) \in \mathbb{R} \times T^* B$.

Note that the assumption (**T1**) and (**T2**) imply that the *Legendre transformation*

$$S^1 \times TB \to S^1 \times T^* B, \qquad (t, q, v) \mapsto (t, q, \nabla^v L(q, v)), . \tag{10.8}$$

is a diffeomorphism. Thus we can make the following definition:

Definition 10.5. The *Fenchel dual Hamiltonian* $H \in C^\infty(S^1 \times T^* B, \mathbb{R})$ is defined by

$$H_t(q, p) := p(v) - L_t(q, v), \qquad \text{where } \nabla^v L_t^(q, v) = p. \tag{10.9}$$

The assumption (**T3**) above is that the flow exists for all $(t, q, p) \in \mathbb{R} \times T^* B$. Note that since $T^* B$ is non-compact, this is not automatically true. However if say, L (and hence H) are autonomous then this *is* automatically satisfied. More generally, we have:

Lemma 10.6. *Suppose the Fenchel dual Hamiltonian H satisfies*

$$\frac{\partial H}{\partial t}(t, q, p) \leq c(1 + H(t, q, p)), \qquad \textit{for all } (t, q, p) \in S^1 \times T^* B. \tag{10.10}$$

Then the flow φ_H^t exists for all $(t, q, p) \in \mathbb{R} \times T^* B$.

Proof. Since $dH_t[X_{H_t}] = 0$, (10.10) implies that

$$\frac{d}{dt} H(t, \varphi_H^t(q,p)) \leq c(1 + H(t, \varphi_H^t(q,p))).$$

Thus by Gronwall's lemma, H is bounded along the flow. Then coercivity of H implies that the flow exists for all time. ■

Exercise 10.7. Show that one has

$$H_t(q,p) = \max_{v \in T_q B} \{p(v) - L_t(q,v)\}.$$

Similarly a Hamiltonian $H \in C^\infty(S^1 \times T^*B, \mathbb{R})$ is called *Tonelli* if the following two conditions are satisfied:

(**T1**') H_t is *fibrewise strictly convex*: that is, the operator $\nabla^{vv} H_t$ is positive definite. Thus for any $(q,p) \in T^*B$, one has

$$\nabla^{vv} H_t(q,p)(z) > 0 \qquad \text{for all } 0 \neq z \in T_q^*B.$$

(**T2**') H_t is *fibrewise superlinear*: that is, for any constant $C > 0$ there exists a finite constant $A(C) > 0$ such that

$$L(t,q,p) \geq C|p| - A(C), \qquad \text{for all } (t,q,p) \in S^1 \times T^*B.$$

(**T3**) The Hamiltonian flow $\varphi_H^t : T^*B \to T^*B$ exists for all $(t,q,p) \in \mathbb{R} \times T^*B$.

As before, the assumption (**T1**') and (**T2**') imply that *Legendre transformation*

$$S^1 \times T^*B \to S^1 \times TB, \qquad (t,q,p) \mapsto (t,q,\nabla^v H_t(q,p)),. \tag{10.11}$$

is a diffeomorphism, and so we can define the *Fenchel dual Lagrangian* L by

$$L_t(q,v) := p(v) - H_t(q,p), \qquad \text{where } \nabla^v H_t(q,p) = v.$$

Exercise 10.8. Show that the Fenchel dual Lagrangian is itself a Tonelli Lagrangian. In fact, show that the two operations $L \mapsto H$ and $H \mapsto L$ invert each other - that is, if L is a Tonelli Lagrangian then the Fenchel dual of the Fenchel dual of L is L again, and similarly with H.

The class of Tonelli Hamiltonians and Lagrangians is the "natural" one to study when working with cotangent bundles. Nevertheless from the point of view of Floer theory it is too difficult to work with *all* Tonelli Hamiltonians. Roughly speaking, this is due to the fact that Tonelli Hamiltonians in general may be too "wild" at infinity. In this course we will work with a slightly different class of Hamiltonians, which I will call Hamiltonians of "quadratic" type. They were introduced by Abbondandolo and Schwarz in their seminal paper [AS06]. Roughly speaking, a Hamiltonian will be of *quadratic type* if H behances like a quadratic form in the p-variables for $|p|$ large. On the one hand, this is a stronger restriction than the Tonelli assumption at infinity. On the other hand, the conditions (**Q1**) and (**Q2**) defined involve only the behaviour of H for $|p|$ large, and they impose no assumptions on compact subsets of T^*B.

Recall that Z denotes the Liouville vector field on T^*B

Definition 10.9. A Hamiltonian $H \in C^\infty(S^1 \times T^*B, \mathbb{R})$ is said to be of *quadratic type* if the following two conditions are satisfied.

(**Q1**) There exist $c_1 > 0$ and $c_2 \geq 0$ such that

$$dH_t(q,p)[Z] - H_t(q,p) \geq c_1|p|^2 - c_2,$$

for every $(t,q,p) \in S^1 \times T^*B$.

(**Q2**) There exists $c_3 \geq 0$ such that

$$|\nabla^h H_t(q,p)| \leq c_3(1+|p|^2), \quad |\nabla^v H_t(q,p)| \leq c_3(1+|p|),$$

for every $(t,q,p) \in S^1 \times T^*B$.

Exercise 10.10. Show that condition (**Q1**) and (**Q2**) imply the existence of constants $c_4, c5 > 0$ such that

$$c_4(1+|p|)^2 \geq H_t(q,p) \geq \frac{1}{2}c_1|p|^2 - c_c, \tag{10.12}$$

and hence conditions (**Q1**) and (**Q2**) really do force H to grow exactly quadratically at infinity.

Exercise 10.11. Show also that condition (**Q1**) does not depend on the choice of the metric on B (up to choosing new constants c_1, c_2).

A slightly more challenging exercise is to show that the condition (**Q2**) also does not depend on the metric:

Exercise 10.12. Show that condition (**Q2**) is equivalent to the following condition:

(**Q2'**) For any set $(q_1, \dots, q_n)$ of local coordinates on B, if $(q_1, \dots, q_n, p_1, \dots, p_n)$ denote the corresponding local coordinates on T^*B, then there exists a constant $C > 0$ (depending on the choice of coordinates) such that

$$\left|\frac{\partial H_t}{\partial q_j}(q,p)\right| \leq C(1+|p|^2), \qquad \left|\frac{\partial H_t}{\partial p_j}(q,p)\right| \leq C(1+|p|), \qquad \text{for all } j = 1, \dots, n. \tag{10.13}$$

Hint: This isn't difficult, but it requires you to go back and carefully examine how $\nabla^h H$ was defined. If you get stuck, see [AS06, p273-274]. Finally, use (10.13) to show that condition (**Q2**) is also independent of the choice of metric on B.

Let us also note that condition (**Q2**) implies that there exists a constant $c_0 > 0$ such that

$$|X_{H_t}(q,p)| = |\nabla H_t(q,p)| \leq c_0(1+|p|^2), \qquad \text{for all } (t,q,p) \in S^1 \times T^*B. \tag{10.14}$$

where here ∇H_t denotes the gradient of H with respect to the Sasaki metric g_{T^*B}.

Remark 10.13. In fact, in all of the arguments that follow, we shall only ever use (10.14), rather than the full strength of condition (**Q2**). The condition (10.14) is strictly weaker than (**Q2**), however it is a somewhat less aesthetically pleasing condition since (10.14) *does* depend on the choice of metric on B. Thus we prefer to stick with the condition (**Q2**).

Lemma 10.14. *Assume that $H \in C^\infty(S^1 \times T^*B)$ is non-degenerate (i.e. all elements $x \in \mathcal{P}_1(H)$ are non-degenerate), and assume that H satisfies both (**Q1**) and (**Q2**). Then for every $a \in \mathbb{R}$, the set of solutions $x \in \mathcal{P}_1(H)$ such that $\mathbb{A}_H(x) \le a$ is finite.*

Proof. Let $x = (q,p) \in \mathcal{P}_1(H)$ be such that $\mathbb{A}_H(x) \le a$. Then by (**Q1**),

$$\begin{aligned} a \ge A_H(x) &= \int_{S^1} (\lambda(\dot{x}) - H_t(x))\, dt = \int_{S^1} (\omega(Z, X_{H_t}(x)) - H_t(x))\, dt \\ &= \int_{S^1} (dH_t(q,p)[Z(q,p)] - H_t(q,p))\, dt \ge c_1 \|p\|^2_{L^2} - c_2, \end{aligned}$$

so $\mathcal{P}_1(H) \cap \{\mathbb{A}_H \le a\}$ is bounded in L^2. By (10.14) we also have

$$|\dot{x}| = |X_H(t,x)| \le c_0(1 + |p|^2),$$

and thus we see that $\mathcal{P}_1(H) \cap \{\mathbb{A}_H \le a\}$ is also bounded in $W^{1,1}$. By the Sobolev embedding theorem, one also has $\mathcal{P}_1(H) \cap \{\mathbb{A}_H \le a\}$ bounded in L^∞. In particular, the set $\{x(0) \mid x \in \mathcal{P}_1(H),\ \mathbb{A}_H(x) \le a\}$ is pre-compact in T^*B. Since H is non-degenerate, this set is also discrete (cf. Exercise 3.28), and thus must be finite. ■

10.5 Grading

As we have already mentioned, when working with cotangent bundles it is convenient to work with *all* one-periodic orbits, rather than just the contractible ones. In Definition 6.22 we explained how to unambiguously associate a Conley-Zehnder index $\mathrm{CZ}(x)$ to any contractible orbit, under the assumption that the map $I_{c_1} : \pi_2(Q) \to \mathbb{Z}$ from Definition 2.22 vanishes. For cotangent bundles, the first Chern class actually zero: $c_1(TB) = 0$ (and thus in particular I_{c_1} also vanishes). We will prove this shortly. Thus we can still use the recipe from Definition 6.22 to define the Conley-Zehnder index for contractible solutions. This will not work for the non-contractible ones though, and hence we need another method. There are various different approaches that one can use; we shall favour the Abbondandolo-Schwarz method of working with *vertical-preserving trivialisations*.

Definition 10.15. Suppose (Q^{2n}, ω) is a symplectic manifold. A *polarisation* of Q is a involutive Lagrangian distribution $\mathcal{L} \subset TQ$Equivalently, a polarisation of Q is a Lagrangian foliation, that is an n-dimensional foliation all of whose leaves are Lagrangian submanifolds of Q.

Lemma 10.16. *Suppose (Q^{2n}, ω) admits a polarisation. Then all the odd Chern classes $c_{2k+1}(TQ)$ are zero.*

Proof. The polarisation allows one to reduce the structure group of TQ from $\mathrm{U}(n)$ to $\mathrm{O}(n)$. ■

Cotangent bundles always admits polarizations: namely the vertical distribution T^vTB. Thus in particular, we see that $c_1(TT^*B) = 0$.

Exercise 10.17. When (if ever) does the horizontal distribution T^hTB define a polarisation?

We will be interested in trivialisations that preserve the vertical distribution.

Definition 10.18. Suppose $x : S^1 \to T^*B$ is a smooth loop. A symplectic trivialisation $\Phi : S^1 \times \mathbb{R}^{2n} \to x^*T(T^*B)$ is called *vertical-preserving* if

$$\Phi(t)(\{0\} \times \mathbb{R}^n) = T^v_{x(t)}T^*B \qquad txforallt \in S^1. \tag{10.15}$$

The following lemma, which is taken from [AS06, Lemma 1.2] shows that such trivialisations always exist. Here it is important that our base manifold B is orientable.

Lemma 10.19. *Assume that B is orientable, and let $x \in \mathcal{P}_1(H)$. Then the symplectic vector bundle $x^*T(T^*B)$ admits a vertical-preserving symplectic trivialisation.*

Proof. Write $x(t) = (q(t), p(t))$. Since B is orientable, the vector bundle $x^*(T^vT^*B)$ is orientable, and hence trivial. Let

$$\Psi : S^1 \times \mathbb{R}^n \to x^*(T^vT^*M)$$

be a trivialisation, and let J be a ω-compatible complex structure on $x^*(TT^*M)$. Then

$$T_{x(t)}T^*B = J(t)T^v_{x(t)}T^*B \oplus T^v_{x(t)}T^*B,$$

and the trivialisation

$$\Phi : S^1 \times \mathbb{R}^n \oplus \mathbb{R}^n \to x^*T(T^*B), \qquad \Phi(t) = (-J(t)\Psi(t)J_0) \oplus \Psi(t),$$

is symplectic and maps $\{0\} \times \mathbb{R}^n$ into the vertical subbundle. ∎

Denote by $\mathrm{Sp}^v(2n, \omega_0)$ the subgroup of the symplectic group consisting of those automorphisms which preserve the vertical Lagrangian subspace $\{0\} \times \mathbb{R}^n$:

$$\mathrm{Sp}^v(2n, \omega_0) := \{W \in \mathrm{Sp}(2n, \omega_0) \mid W(\{0\} \times \mathbb{R}^n) = \{0\} \times \mathbb{R}^n\}.$$

The subgroup $\mathrm{Sp}^v(2n, \omega_0)$ retracts onto its closed subgroup

$$\mathrm{Sp}^v(2n, \omega_0) \cap U(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \mid R \in \mathrm{O}(n) \right\}.$$

The determinant map $\det : \mathrm{U}(n) \to S^1$ takes the values $\{\pm 1\}$. It follows that $\mathrm{Sp}^v(2n, \omega_0)$ and $\mathrm{Sp}^v(2n, \omega_0) \cap \mathrm{U}(n)$ have two connected components, and that the inclusions

$$\mathrm{Sp}^v(2n, \omega_0) \hookrightarrow \mathrm{Sp}(2n, \omega_0), \qquad \mathrm{Sp}^v(2n, \omega_0) \cap \mathrm{U}(n) \hookrightarrow \mathrm{U}(n),$$

induce the zero homomorphism between fundamental groups.

Lemma 10.20. *If $x \in \mathcal{P}_1(H)$ and Φ_1 and Φ_2 are two vertical-preserving trivialisations, and we define symplectic paths $\Psi_j : [0, 1] \to \mathrm{Sp}(2n, \omega_0)$*

$$\Psi_j(t) := \Phi_{j,t}^{-1} \circ D\varphi_H^t(x(0)) \circ \Phi_{j,0}, \qquad j = 1, 2,$$

then the two Conley-Zehnder indices agree:

$$\mathrm{CZ}(\Psi_1) = \mathrm{CZ}(\Psi_2).$$

Proof. We can write

$$\Psi_1(t) = W(t)\Psi_2(t)W(0)^{-1},$$

for some

$$W : S^1 \to \mathrm{Sp}^v(2n, \omega_0).$$

Since the inclusion $\mathrm{Sp}^v(2n, \omega_0) \hookrightarrow \mathrm{Sp}^v(2n, \omega_0)$ induces the zero homomorphism between fundamental groups, $W(0)\Psi_2(t)W(0)^{-1}$ and Ψ_1 are homotopic by a homotopy which fixes the end-points. The homotopy and the naturality property of the Conley-Zehnder index (see Theorem 6.7) index imply that $\mathrm{CZ}(\Psi_1) = \mathrm{CZ}(\Psi_2)$. ∎

Thus for cotangent bundles we can given the following definition.

Definition 10.21. The Conley-Zehnder index of a periodic solution $x \in \mathcal{P}_1(H)$ is the integer $\mathrm{CZ}(x) := \mathrm{CZ}(\Psi)$, where Ψ is a the symplectic path (as in (6.11)) associated to some (and hence any) vertical-preserving trivialisation Φ of of $x^*T(T^*B)$.

Exercise 10.22. Suppose that $x \in \mathcal{P}_1^\circ(H)$ is a contractible periodic orbit. Then we now have two different ways to define a Conley-Zehnder index: either by choosing a trivialisation that comes from a admissible trivialisation in the sense of Definition 3.18, or by choosing a vertical-preserving trivialisation, as in Definition 10.21. Show that these always give the same answer, that is, the Conley-Zehnder index associated to an admissible trivialisation always agrees with the Conley-Zehnder index associated to a vertical-preserving trivialisation.

We also need to trivialise pullback bundles coming from flow lines. For this it is important to know that if u is a flow line running from x^- to x^+, and $\Phi^\pm$ are two vertical-preserving trivialisations of $(x^\pm)^*T(T^*B)$ then we can find a vertical-preserving trivialisation Φ of $u^*T(T^*B)$ that restricts to $\Phi^\pm$ over the ends. The next lemma, which is taken from [AS06, Lemma 1.7] makes this precise. Recall that given a flow line $u \in \mathcal{M}(x^-, x^+)$ we denote by $\overline{u}$ the compactified map $\overline{\mathbb{R}} \times S^1 \to T^*B$ (cf. Exercise 5.3).

Lemma 10.23. *Let $u \in \mathcal{M}(x^-, x^+)$, and let*

$$\Phi^\pm : S^1 \times \mathbb{R}^{2n} \to x^{\pm *}T(T^*B)$$

be two vertical-preserving symplectic trivialisations. Then there exists a vertical-preserving symplectic trivialisation

$$\Phi : \overline{\mathbb{R}} \times S^1 \times \mathbb{R}^{2n} \to u^*T(T^*B)$$

such that $\Phi(\pm\infty, t) = \Phi^\pm(t)$ for every $t \in S^1$.

Proof. By the same construction used in the proof of Lemma 10.20, we can find a certainly find *some* vertical-preserving trivialisation

$$\Phi_0 : \overline{\mathbb{R}} \times S^1 \times \mathbb{R}^{2n} \to u^*T(T^*B).$$

However this Φ_0 will no have the correct asymptotics. Consider the loops in $\mathrm{Sp}^v(2n, \omega_0) \cap \mathrm{U}(n)$

$$W^\pm(t) := \Phi_0(\pm\infty, t)^{-1}\Phi^\pm(t).$$

Since the inclusion $\mathrm{Sp}^v(2n, \omega_0) \cap \mathrm{U}(n) \hookrightarrow U(n)$ induces the zero homomorphism between fundamental groups, and since $\mathrm{U}(n)$ is connected, we can find a homotopy

$$W : \overline{\mathbb{R}} \times S^1 \to \mathrm{U}(n)$$

such that $W(\pm\infty, t) = W^\pm(t)$ for every $t \in S^1$. Then the unitary trivialisation

$$\Phi(s,t) := \Phi_0(s,t)W(s,t), \qquad (s,t) \in \overline{\mathbb{R}} \times S^1,$$

has the required asymptotics. ■

10.6 The L^∞-estimates

Since T^*B is non-compact, there is an extra difficulty in proving the compactness arguments needed to define the boundary operator. In this case the space of all finite energy gradient flow lines of $\mathbb{A}_H$ is *not* compact, that is, the analogue of Theorem 8.8 is false. Nevertheless if we restrict to a finite action window we do have bounds. This is the content of the following result, which in the form we present, is due to Abbondandolo and Schwarz [AS06, Theorem 1.14].

Theorem 10.24 (The L^∞-estimates)**.** *Assume H satisfies (**Q1**) and (**Q2**). Fix $\alpha, \beta \in \mathbb{R}$, and let $\mathcal{M}_{\alpha,\beta}$ denote the space of all gradient flow lines of $\mathbb{A}_H$ that satisfy*

$$\alpha \leq \mathbb{A}_H(u(s,\cdot)) \leq \beta, \qquad \text{for all } s \in \mathbb{R},$$

Then $\mathcal{M}_{\alpha,\beta}$ is bounded in $L^\infty(\mathbb{R} \times S^1, T^*B)$.

The proof of Theorem 10.24 has two distinct parts. The first part uses crucially the fact that H satisfies (**Q1**) and (**Q2**), and obtains $W^{1,2}$-bounds on compact intervals. The second stage makes use of the fact that we have embedded T^*B into $\mathbb{R}^{2N}$ in such a way that the almost complex structure J_g is the restriction of the canonical one J_0.

Theorem 10.25. *There exists a constant $c = c(\alpha, \beta, H)$ such that if $u = (q,p) \in \mathcal{M}_{\alpha,\beta}$ and $I \subset \mathbb{R}$ is a compact interval then one has*

$$\|p\|_{L^2(I\times S^1)} \leq c\sqrt{|I|}, \tag{10.16}$$

$$\|\nabla p\|_{L^2(I\times S^1)} \leq c(\sqrt{|I|}+1). \tag{10.17}$$

Proof. We will prove (10.16), and indicate how the argument extends to establish (10.17), referring the reader to [AS06, Lemma 1.12] for the rest of the details. We will prove the claim in seven stages.
Step 1 Let us first show note that for every $u \in \mathcal{M}_{\alpha,\beta}$ one has

$$\|\partial_s u\|_{L^2(\mathbb{R}\times S^1)} \leq a_1 \tag{10.18}$$

To show this take $s_0 < s_1$ and compute

$$\|\partial_s u\|_{L^2((s_0,s_1)\times S^1)} = \int_{s_0}^{s_1}\int_{S^1} |\partial_s u|^2 dtds \leq \mathbb{A}_H(u(s_0,\cdot)) - \mathbb{A}_H(u(s_1,\cdot)).$$

Thus (10.18) follows with $a_1 := \beta - \alpha$.
Step 2 Now let us show that there exists a constant $a_2 > 0$ such that

$$\|p(s,\cdot)\|_{L^2(S^1)} \leq a_2(1 + \|\partial_s u(s,\cdot))\|_{L^2(S^1)}. \tag{10.19}$$

To see this we first observe that

$$\begin{aligned}\lambda(\partial_t u) &= \lambda(J_g\partial_s u + X_{H_t}(u)) \\ &= \omega(Z(u), J_g\partial_s u + X_{H_t}(u)) \\ &= dH_t(u)[Z(u)] - g_{T^*B}(Z(u), \partial_s u),\end{aligned}$$

and hence using the fact that $|Z(q,p)| = |p|$ and condition (**Q1**), we see that

$$\lambda(\partial_t u) - H_t(u) \geq c_1|p|^2 - c_2 - |p||\partial_s u|. \tag{10.20}$$

Integrating this equation over S^1 we obtain

$$\beta \geq \mathbb{A}_H(u(s,\cdot)) = \int_{S^1} \lambda(\partial_t u(s,t)) - H_t(u(s,t))dt$$
$$\geq c_1 \|p(s,\cdot)\|^2_{L^2(S^1)} - c_2 - \|p(s,\cdot)\|_{L^2(S^1)} \|\partial_s u\|_{L^2(S^1)},$$

which implies (10.19).

Step 3 Next we show there exists a_3 such that

$$\|p(s,\cdot)\|_{L^\infty(S^1)} \leq a_3(1 + \|\partial_s u(s,\cdot)\|^2_{L^2(S^1)}), \qquad \text{for every } s \in \mathbb{R}. \tag{10.21}$$

Indeed, by (10.14),

$$\|\partial_t p(s,\cdot)\|_{L^1(S^1)} \leq \|\partial_t u(s,\cdot)\|_{L^1(S^1)}$$
$$\leq \|X_H(\cdot, u(s,\cdot))\|_{L^1(S^1)} + \|J_g \partial_s u(s,\cdot)\|_{L^1(S^1)}$$
$$\leq c_0(1 + \|p(s,\cdot)\|^2_{L^2(S^1)}) + \|\partial_s u(s,\cdot)\|_{L^1(S^1)},$$

where we have used the fact that the almost complex structure J_g has norm 1 (with respect to the Sasaki metric g_{T^*B}). Then by (10.19), have

$$\|\partial_t p(s,\cdot)\|_{L^1(S^1)} \leq c_0(1 + a_2^2(1 + \|\partial_s u(s,\cdot)\|_{L^2(S^1)})^2) + \|\partial_s u(s,\cdot)\|_{L^2(S^1)}.$$

Thus the $W^{1,1}$ norm of $p(s,\cdot)$ on S^1 is bounded by a quadratic function of $\|\partial_s u(s,\cdot)\|_{L^2(S^1)}$. The Sobolev embedding theorem implies the same for the L^∞ norm, which completes the proof of (10.21).

Step 4 We show that for every $\varepsilon > 0$ there exists a number $R(\varepsilon) > 0$ such that if $u \in \mathcal{M}_{\alpha,\beta}$ and

$$S(u;\varepsilon) := \left\{ s \in \mathbb{R} \mid \|p(s,\cdot)\|_{L^\infty(S^1)} \right\}$$

then if I is any interval of $\mathbb{R}$ with $|I| \geq \varepsilon$ then $S(u;\varepsilon) \cap I \neq \emptyset$. To see this fix $s_0 \in \mathbb{R}$ and observe that

$$\min_{s \in [s_0, s_0+\varepsilon]} \|\partial_s u(s,\cdot)\|^2_{L^2(S^1)} \leq \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} \|\partial_s u(s,\cdot)\|^2_{L^2(S^1)} ds$$
$$= \frac{1}{\varepsilon} \|\partial_s u\|^2_{L^2([s_0, s_0+\varepsilon] \times S^1)}$$
$$\leq \frac{a_1^2}{\varepsilon}.$$

Thus Step 3 implies the result with

$$R(\varepsilon) := a_3 \left(1 + \frac{a_1^2}{\varepsilon} \right).$$

Step 4.5 We insert a random meaningless line of text to see whether anyone notices.

Step 5 We show there exists a constant $a_4 > 0$ such that $\|p(s,\cdot)\|_{L^\infty(S^1)} \leq a_4$ for all $s \in \mathbb{R}$. Indeed, fix $s \in \mathbb{R}$ and take $s_0 \in S(u;1)$ such that $|s - s_0| \leq 1$. Let us assume for simplicity that $s_0 \leq s$. Then we have

$$\|p(s,\cdot)\|^2_{L^2(S^1)} = \|p(s_0,\cdot)\|_{L^2(S^1)} + \int_{s_0}^{s} \frac{d}{\tau} \|p(\tau,\cdot)\|^2_{L^2(S^1)} d\tau$$
$$= \|p(s_0,\cdot)\|_{L^2(S^1)} + 2 \int_{s_0}^{s} \int_{S^1} \langle p(\tau,t), \partial_\tau p(\tau,t) \rangle dt\, d\tau$$
$$\leq R(1)^2 + 2 \left| \int_{s_0}^{s} \|p(\tau,\cdot)\|^2_{L^2(S^1)} d\tau \right|^{1/2} \|\partial_s p\|_{L^2([s_0,s] \times S^1)}.$$

Since $\|\partial_s p\|_{L^2([s_0,s]\times S^1)} \le \|\partial_s u\|_{L^2([s_0,s]\times S^1)} \le a_1$ by Step 1, we see that

$$\begin{aligned}\|p(s,\cdot)\|^2_{L^2(S^1)} &\le R(1)^2 + 2a_1 \left| \int_{s_0}^{s} a_2^2 (1 + \|\partial_\tau u(\tau,\cdot)\|^2_{L^2(S^1)} d\tau \right|^{1/2} \\ &\le R(1)^2 + 2a_1 a_2 (2|s-s_0| + 2\|\partial_s u\|^2_{L^2([s_0,s]\times S^1)})^{1/2} \\ &\le R(1)^2 + 2a_1 a_2 (2 + 2a_1^2)^{1/2}.\end{aligned}$$

Step 6 Given $u = (q,p) \in \mathcal{M}_{\alpha,\beta}$ and $I \subset \mathbb{R}$ a compact interval, we have by Step 5 that

$$\|p\|^2_{L^2(I\times S^1)} = \int_I \|p(s,\cdot)\|^2_{L^2(S^1)} ds \le a_4^2 |I|.$$

Step 7 We now briefly indicate how to prove (10.17). Note that by (10.14) we have

$$\begin{aligned}|\nabla p|^2 &\le |\nabla u|^2 \\ &= |\partial_s u|^2 + |\partial_t u|^2 \\ &= |\partial_s u|^2 + |J_g(u)\partial_s u + X_{H_t}(u)|^2 \\ &\le 3|\partial_s u|^2 + 2c_0^2(1+|p|^2)^2.\end{aligned}$$

Thus we see that in order to deal with $\|\nabla p\|_{L^2(I\times S^1)}$, we need to be able to cope with the L^4-norm of p, $\|p\|_{L^4(I\times S^1)}$. This requires an *interpolation* estimate:

Lemma 10.26. *There exists a $C > 0$ such that for every $f \in W^{1,2}(\mathbb{R}\times S^1)$ one has*

$$\|f\|^4_{L^4(\mathbb{R}\times S^1)} \le C \|f\|^2_{L^2(\mathbb{R}\times S^1)} \|f\|^2_{W^{1,2}(\mathbb{R}\times S^1)}.$$

A proof of the lemma can be found in [AS06, Lemma 1.11]. The rest of the proof can be found in [AS06, p278-279]. ■

We now use Theorem 10.25 to prove Theorem 10.24.

Proof of Theorem 10.24. Fix a smooth cutoff function $\beta : \mathbb{R} to [0,1]$ such that $\beta(s) = 1$ for $s \in [0,1]$, and $\beta(s) = 0$ for s outside of $[-1,2]$, and such that $\|\beta'(s)\| \le 2$ for all $s \in \mathbb{R}$. Given $k \in \mathbb{Z}$, let

$$v_k : \mathbb{R}\times S^1 \to T^*B, \qquad v_k(s,t) := \beta(s-k)u(s,t).$$

Then with $\bar\partial := \partial_s + J_0 \partial_t$ as usual, we have

$$\bar\partial v_k = \beta'(s)u + \beta J_0 X_{H_t}(u)$$

Fix $r > 2$ (for instance, $r = 3$ works). Since v_k is compactly supported, we can use the Calderon-Zygmund inequality (Exercise 4.7) to estimate

$$\begin{aligned}\|\nabla v_k\|_{L^r(\mathbb{R}\times S^1)} &\le c(r)\|\bar\partial v_k\|_{L^r(\mathbb{R}\times S^1)} \\ &\le 4\cdot 3^{1/r} c(r) d + 2\|p\|_{L^r([k-1,k+2]\times S^1)} + c(r)\|X_{H_t}(u)\|_{L^r([k-1,k+2]\times S^1)},\end{aligned} \tag{10.22}$$

where d is the diameter of B inside $\mathbb{R}^N$, and $c(r)$ is the constant provided by the Calderon-Zygmund inequality. Let $K(r)$ denote the norm of the continuous embedding

$$W^{1,2}([0,3]\times S^1) \hookrightarrow L^r([0,3]\times S^1).$$

Using Theorem 10.25, we see that

$$\begin{aligned}\|p\|_{L^r([k-1,k+2]\times S^1)} &\le K_r\|p\|_{W^{1,2}([k-1,k+2]\times S^1)}\\ &\le K_r c(3+(1+\sqrt{3})^2)^{1/2}\\ &\le 4K_r c.\end{aligned} \tag{10.23}$$

Similarly using (10.14) and (10.23), we see that

$$\begin{aligned}\|X_{H_t}(u)\|_{L^r([k-1,k+2]\times S^1)} &\le c_0(3^{1/r}+\|p\|^2_{L^{2r}([k-1,k+2]\times S^1)})\\ &\le c_0(3^{1/r}+K_{2r}^2\|p\|^2_{W^{1,2}([k-1,k+2]\times S^1)})\\ &\le c_0(3^{1/r}+16K_{2r}^2c^2).\end{aligned} \tag{10.24}$$

Combining (10.22), (10.23) and (10.24), we obtain a uniform bound on ∇v_k in $L^r(\mathbb{R}\times S^1)$. Thus u is uniformly bounded in $W^{1,r}([k,k+1]\times S^1)$, and since $r>2$, also in $L^\infty([k,k+1]\times S^1)$. Finally, since k was arbitrary, we obtain a uniform bound for u in $L^\infty(\mathbb{R}\times S^1)$. This completes the proof. ∎

There is one more thing to say about compactness before we can move on to define the Floer complex for cotangent bundles. Namely, when it comes to proving invariance, we will need an analogous L^∞-estimate for solutions of an appropriate s-dependent problem. This is the content of the following result, which is taken from [AS06, Lemma 1.21].

Let $\zeta:\mathbb{R}\to[0,1]$ denote a smooth function such that $\zeta(s)=0$ for $s\le 0$ and $\zeta(s)=1$ for $s\ge 1$, with $0\le\zeta'(s)\le 2$ for all $s\in\mathbb{R}$. Suppose H_0 and H_1 are two Hamiltonians satisfying (**Q1**) and (**Q2**), and define

$$H:\mathbb{R}\times S^1\times T^*B\to\mathbb{R},\qquad H(s,t,x):=\zeta(s)H_1(t,x)+(1-\zeta(s))H_0(t,x).$$

Given $\alpha\le\beta$, let $\mathcal{N}_{\alpha,\beta}$ denote the space of all smooth maps $u\in C^\infty(\mathbb{R}\times S^1,T^*B)$ solving

$$\partial_s u+J_g(u)(\partial_t u-X_{H_{s,t}}(u))=0$$

and such that

$$\mathbb{A}_{H_0}(u(s,\cdot))\le\beta,\quad \forall\, sle0,\qquad \mathbb{A}_{H_1}(u(s,\cdot))\ge\alpha,\quad\forall\, s\ge 1.$$

Theorem 10.27. *There exists a constant $\delta(c_1,c_2,c_3)>0$ such that if H_0 and H_1 are two Hamiltonians of quadratic type that satisfy (**Q1**) and (**Q2**) with the same constants c_1,c_2,c_3 and in addition satisfy*

$$|H_0(t,q,p)-H_1(t,q,p)|\le b+\delta|p|^2,\qquad \textit{for all } (t,q,p)\in S^1\times T^*B \tag{10.25}$$

for some constants $b\ge 0$ and some $0<\delta<\delta(c_1)$, then: given any $\alpha\le\beta$, the space $\mathcal{N}_{\alpha,\beta}$ is bounded in $L^\infty(\mathbb{R}\times S^1)$.

Proof. We prove the result in three steps.

Step 1 We first show that for any $u\in\mathcal{N}_{\alpha,\beta}$, one has

$$\mathbb{A}_{H_s}(u(s,\cdot))\le\beta+2b+2\delta\|p\|^2_{L^2((0,1)\times S^1)}. \tag{10.26}$$

and

$$\mathbb{E}(u)=\|\partial_s u\|^2_{L^2(\mathbb{R}\times S^1)}\le\beta-\alpha+2b+2\delta\|p\|^2_{L^2((0,1)\times S^1)}. \tag{10.27}$$

Indeed, to prove (10.26), it is sufficient to prove the result for $s \in [0,1]$, since

$$\begin{aligned} \mathbb{A}_{H_s}(u(s,\cdot)) = \mathbb{A}_{H_0}(u(s,\cdot)) \le \beta, \qquad & \text{for all } s \le 0, \\ \mathbb{A}_{H_s}(u(s,\cdot)) = \mathbb{A}_{H_1}(u(s,\cdot)) \le \mathbb{A}_{H_1}(u(1,\cdot)), \qquad & \text{for all } s \ge 1. \end{aligned}$$

For $s \in [0,1]$, we estimate

$$\begin{aligned} \mathbb{A}_{H_s}(u(s,\cdot)) &= \mathbb{A}_{H_0}(u(0,\cdot)) + \int_0^s \frac{d}{dr}\left(\mathbb{A}_{H_r}(u(r,\cdot))\right) dr \\ &= \beta - \int_0^s \int_{S^1} |\partial_s u|_J^2 dt dr + \int_0^s \zeta'(r) \int_{S^1} (H_0(t,u) - H_1(t,u)) dt dr \\ &\le \beta + 2(bs + \delta \int_0^s \|p(r,\cdot))\|^2_{L^2(S^1)} dr \\ &\le \beta + 2b + 2\delta \|p\|^2_{L^2((0,1)\times S^1)}. \end{aligned}$$

The proof of (10.27) is similar.

Step 2 We show that for all $u \in \mathcal{N}_{\alpha,\beta}$, one has

$$\frac{c_1}{2}\|p(s,\cdot)\|^2_{L^2(S^1)} - (c_2 + \beta + 2b) \le 2\delta \|p\|^2_{L^2((0,1)\times S^1)} + \frac{1}{2c_1}\|\partial_s u(s,\cdot)\|^2_{L^2(S^1))}. \tag{10.28}$$

To see this we first argue as in the proof of (10.20) to see that

$$\lambda(\partial_t u) - H(s,t,u) \ge c_1 |p|^2 - c_2 - |p||\partial_s u|,$$

and hence by (10.26) we see

$$\begin{aligned} \beta + 2b + 2\delta \|p\|^2_{L^2((0,1)\times S^1)} &\ge \mathbb{A}_{H_s}(u(s,\cdot)) \\ &\ge c_1 \|p(s,\cdot)\|^2_{L^2(S^1)} - c_2 - \|p(s,\cdot)\|_{L^2(S^1)} \|\partial_s u(s,\cdot)\|_{L^2(S^1)} \\ &\ge \frac{c_1}{2}\|p(s,\cdot)\|^2_{L^2(S^1)} - c_2 - \frac{1}{2c_1}\|\partial_s u(s,\cdot)\|^2_{L^2(S^1)}, \end{aligned}$$

which implies (10.28).

Step 3 We now integrate (10.28) over $(0,1)$ and use (10.27) to see that

$$\begin{aligned} \frac{c_1}{2}\|p\|^2_{L^2((0,1)\times S^1)} - (c_2 + \beta + 2b) &\le 2\delta \|p\|^2_{L^2((0,1)\times S^1)} + \frac{1}{2c_1}\|\partial_s u\|^2_{L^2((0,1)\times S^1)} \\ &\le 2\delta \|p\|^2_{L^2((0,1)\times S^1)} + \frac{1}{2c_1}\left(\beta - \alpha + 2b + 2\delta\|p\|^2_{L^2((0,1)\times S^1)}\right). \end{aligned}$$

In other words,

$$\left(\frac{c_1}{2} - 2\delta\left(1 + \frac{1}{2c_1}\right)\right)\|p\|^2_{L^2((0,1)\times S^1)} \le c_2 + \beta + 2b + \frac{1}{2c_1}(\beta - \alpha + 2b).$$

This tells us that if

$$0 < \delta < \delta(c_1, c_2) := \frac{c_1}{4}\left(1 + \frac{1}{2c_1}\right)^{-1}$$

then $\|p\|_{L^2((0,1)\times S^1)}$ is uniformly bounded for $u = (q,p) \in \mathcal{N}_{\alpha,\beta}$. Thus we can improve (10.26) and (10.27) to the claim that there exist constants $a_1', a_2' > 0$ such that

$$\begin{aligned} &\|\partial_s u\|_{L^2(\mathbb{R}\times S^1)} \le a_1', \\ &\|p(s,\cdot)\|_{L^2(S^1)} \le a_2'(1 + \|\partial_s u(s,\cdot))\|_{L^2(S^1)}. \end{aligned}$$

These two equations were exactly the conclusions of Steps 1 and 2 of the proof of Theorem 10.25. The reader may now easily check that the rest of the proof of Theorem 10.25 goes through without change. Moreover the proof of Theorem 10.24 goes through without any changes. This completes the proof. ■

We can now define the Floer homology $\mathrm{HF}_*(H)$ and prove that it is independent of the choice of Hamiltonian H satisfying (**Q1**) and (**Q2**). Since we know that solutions to the Floer equation with bounded action remain in a compact subset of T^*B, the transversality and compactness analysis we performed earlier go through without any changes. Thus from now on let us assume that H is a Hamiltonian satisfying (**Q1**) and (**Q2**), such that additionally:

1. Every element of $\mathcal{P}_1(H)$ is non-degenerate.
2. The linearization $D\overline{\partial}_{J_g,H}(u)$ at every solution u of the Floer equation for (J_g, H) admits a left inverse, and hence the moduli spaces $\mathcal{M}(x^-, x^+)$ for $x^\pm \in \mathcal{P}_1(H)$ are all manifolds.

Definition 10.28. We define $\mathrm{CF}_*(H)$ to be the graded $\mathbb{Z}_2$-vector space generated by all formal sums

$$\sum_{x\in\mathcal{X}} \langle x \rangle ,$$

where $\mathcal{X} \subset \mathcal{P}_1(H)$ is a subset such that

$$\sup_{x\in\mathcal{X}} \mathbb{A}_H(x) < +\infty.$$

The grading is given by the Conley-Zehnder index. Note that such a set $\mathcal{X}$ is necessarily finite, thanks to Lemma 10.14.

The boundary operator is defined in the standard way:

$$\partial \langle x \rangle := \sum_{\substack{y\in\mathcal{P}_1(H) \\ \mathrm{CZ}(y)=\mathrm{CZ}(x)-1}} n(x,y) \langle y \rangle ,$$

$$n(x,y) := \#_2 \underline{\mathcal{M}}(x,y).$$

The fact that $\partial \langle x \rangle$ is a well defined element of $\mathrm{CF}_*(H)$ follows from the fact that $n(x,y) = 0$ if $\mathbb{A}_H(y) > \mathbb{A}_H(x)$, together with Lemma 10.14.

Now we explain how Theorem 10.27 can be used to show that the resulting homology $\mathrm{HF}_*(H)$ is independent of the choice of Hamiltonian H satisfying (**Q1**) and (**Q2**). Indeed, suppose H_0 and H_1 are two such Hamiltonians. We may assume that they both satisfy (**Q1**) and (**Q2**) with the same constants c_1, c_2, c_3. By (10.12), there exists a constant $c_4 > 0$ such that

$$|H_0(t,q,p)| \le c_4(1+|p|^2), \qquad |H_1(t,q,p)| \le c_4(1+|p|^2). \tag{10.29}$$

Let $\sigma \in [0,1]$, and define $H_\sigma := \sigma H_1 + (1-\sigma)H_0$. Given $\sigma_0, \sigma_1 \in [0,1]$, we see from (10.29) that

$$|H_{\sigma_1}(t,q,p) - H_{\sigma_0}(t,q,p)| \le 2c_4|\sigma_1 - \sigma_0|(1+|p|^2).$$

In particular, if $|\sigma_1 - \sigma_0| < \delta(c_1)/2c_4$, where $\delta(c_1)$ was specified in Theorem 10.27, then we have L^∞-estimates for the s-dependent problem interpolating between H_{σ_0} and H_{σ_1}. Then the analogue of Theorem 9.3 tells us that there exists a well-defined chain map

$$\Phi_{\sigma_0,\sigma_1} : \mathrm{CF}_*(H_{\sigma_1}) \to \mathrm{CF}_*(H_{\sigma_1})$$

inducing an isomorphism ϕ_{σ_0,σ_1} on homology, which moreover is suitably functorial (in the sense specified by Theorem 9.3). We now choose a sequence

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = 1, \tag{10.30}$$

such that $0 < \sigma_j - \sigma_{j-1} < \delta(c_1)/2c_4$ for each $j = 1, \dots, N$, and such that each H_{σ_j} is non-degenerate in the sense described above.

Exercise 10.29. Show that such a sequence (10.30) exists.

One finally defines the map

$$\phi : \mathrm{HF}_*(H_0) \to \mathrm{HF}_*(H_1), \qquad \phi := \phi_{\sigma_{N-1},\sigma_N} \circ c \cdots \circ \phi_{\sigma_0,\sigma_1}.$$

We emphasise that in general there is no *chain map* between $\mathrm{CF}_*(H_0)$ and $\mathrm{CF}_*(H_1)$. The idea of breaking an homotopy down into many little pieces and then concatenating the corresponding maps is called the *adiabatic method*.

CHAPTER 11

Infinite dimensional Morse homology

At this point I ran out of time...

CHAPTER 12

The Abbondandolo-Schwarz isomorphism

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